# Homology in Semi-Abelian Categories (full version)

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I have used Dr Julia Goedecke's thesis [1] as my main source of information. In addition, many other books and papers are used in chapter 2. Indeed, [7] was helpful for sections 2.1 and 2.3 and I understood results in sections 2.4 and 2.5 thanks to [3] and [4]. I also used [3] in section 2.6. The proof of lemma 2.43 in section 2.7 is based on [9]. Chapters 3 and 4 were largely inspired from [1]. However, I used [10] in section 3.1 and [6] in section 3.2. Examples in section 4.3 come from [5] and [11].

# 1 Introduction

This essay is about comonadic homology in semi-abelian categories.

Homological algebra appears in every area of Algebra: Group Theory, Commutative Algebra, Algebraic Geometry, Algebraic Topology, and so forth. It can be used to measure the exactness of a chain complex. In Category Theory, we defined homology in abelian categories, like the category AbGp of abelian groups or $R$-Mod of $R$-modules. Unfortunately, the category of groups and the one of Lie algebras are not abelian.

In order to encompass those examples, semi-abelian categories are defined. Of course abelian categories are semi-abelian, but Gp and LieAlg are semi-abelian as well. Semi-abelian categories are defined as regular (to have an image factorisation) and pointed (to be able to define kernels and cokernels). Moreover, in order to define homology in those categories, they are required to be Barr exact, to have binary coproducts and that the regular Short Five Lemma holds. We can prove (see chapter 2) that they are Mal'cev, finitely cocomplete and that every regular epimorphism is normal. In semi-abelian categories, homology is defined in a similar way to abelian ones, but we need to assume that the chain complex is proper (see section 2.7).

A fruitful way to construct a proper chain complex is to introduce simplicial objects. In chapter 3, we define a simplicial object in any category as a diagram of the form

$$\begin{tikzcd}
\cdots \arrow[r, shift left=4.5] \arrow[r, shift left=1.5] \arrow[r, shift right=1.5] \arrow[r, shift right=4.5] & A_2 \arrow[l, bend left=30] \arrow[l, bend left=45] \arrow[l, bend left=60] \arrow[r, shift left=3] \arrow[r] \arrow[r, shift right=3] & A_1 \arrow[l, bend left=30] \arrow[l, bend left=45] \arrow[r, shift left=1.5] \arrow[r, shift right=1.5] & A_0 \arrow[l, bend left=30]
\end{tikzcd}$$

which satisfies some equalities called simplicial identities. If we work in a semi-abelian category, such objects induce a proper chain complex, and so a homology. This homology is studied in chapter 3.

A particular kind of simplicial object is the one arising from a comonad. Indeed, if $\mathbb{G}$ is a comonad in an arbitrary category $\mathcal{D}$, it induces a simplicial object $\mathbb{G}A$ for each object $A$ in $\mathcal{D}$. To be able to compute its homology, we have to 'push' it forwards into a semi-abelian category $\mathcal{A}$. It can be done thanks to a functor $E : \mathcal{D} \to \mathcal{A}$. The induced homology is then called a comonadic homology and is denoted by $H_n(A, E)_{\mathbb{G}}$. A natural issue about this homology is to know what happens if the parameters $A$, $E$ or $\mathbb{G}$ change. We shall see that it is actually a functor in $A$ and $E$. But what about the third parameter? The main goal of this essay will be to prove that $H_n(A, E)_{\mathbb{G}}$ and $H_n(A, E)_{\mathbb{K}}$ are naturally isomorphic if $\mathbb{G}$ and $\mathbb{K}$ generate the same Kan projective class (theorem 4.10). The main part of the work will be accomplished in chapter 3 where we shall prove a Comparison Theorem (3.21) to create and compare maps between simplicial objects.

Fortunately, this homology is really useful in almost all algebraic subject of Mathematics. Indeed, many well-known homology theories come from a comonadic homology, e.g. Tor and Ext functors in Commutative Algebra, singular and simplicial homologies in Algebraic Topology, integral group homology in Group Theory, and so forth.

Note that in this essay, we are using the notation $\mathbb{N}$ for the set $\{0, 1, \dots\}$ of natural numbers, while $\mathbb{N}_0$ is the set $\{1, 2, \dots\}$.

# 2 Semi-Abelian Categories

As announced in the introduction, we are going to work in semi-abelian categories. In this chapter, we define this notion and state its first few properties. In the last two sections of this chapter, we define exact sequences and their homology in a semi-abelian category, which will be useful in chapters 3 and 4.

## 2.1 Relations

In this section, we define the notion of a relation in a finitely complete category. We shall need it to define exact and semi-abelian categories. It will be clear that the following definitions are generalizations of the concept of relations as we know it in the category of sets.

**Definition 2.1.** Let $\mathcal{C}$ be a finitely complete category and $X, Y \in \text{ob } \mathcal{C}$. A **relation** $R$ between $X$ and $Y$ is the data of two morphisms $X \xleftarrow{d_0} R \xrightarrow{d_1} Y$ in $\mathcal{C}$ such that the induced map $R \xrightarrow{(d_0,d_1)} X \times Y$ is a monomorphism. A relation is said to be **internal** if $X = Y$.

**Example 2.2.** If $\mathcal{C} = \text{Set}$, $(d_0, d_1)$ is a monomorphism if and only if $R$ is a subset of $X \times Y$, i.e. $R$ is a relation (in the usual sense) between $X$ and $Y$.

**Definition 2.3.** Let $\mathcal{C}$ be a finitely complete category and $R \underset{d_1}{\overset{d_0}{\rightrightarrows}} X$ a (internal) relation in $\mathcal{C}$. We say that $R$ is **reflexive** if $X \xrightarrow{(1_X,1_X)} X \times X$ factors through $R \xrightarrow{(d_0,d_1)} X \times X$.

**Example 2.4.** If we go back to our example (i.e. $\mathcal{C} = \text{Set}$), it is equivalent to the 'usual' definition of reflexivity. Indeed, $R$ is reflexive if and only if there is a function $X \xrightarrow{h} R$ such that $d_0 h = d_1 h = 1_X$. Or, equivalently, if and only if for all $x \in X$, there is a $r \in R$ such that $(d_0(r), d_1(r)) = (x, x)$.

**Definition 2.5.** Let $\mathcal{C}$ be a finitely complete category and $R \underset{d_1}{\overset{d_0}{\rightrightarrows}} X$ a relation in $\mathcal{C}$. $R$ is said to be **symmetric** if there exists a morphism $R \xrightarrow{\sigma} R$ such that $d_0 \circ \sigma = d_1$ and $d_1 \circ \sigma = d_0$.

**Example 2.6.** Again, if $\mathcal{C} = \text{Set}$, there is no difference between the usual notion of symmetric relation and the categorical one. Indeed, the existence of such a $\sigma$ is equivalent to the the existence, for all couples $(d_0(r), d_1(r))$ in the relation, of an $r' \in R$ such that $(d_1(r), d_0(r)) = (d_0(r'), d_1(r'))$, which is in the relation.

**Definition 2.7.** Let $\mathcal{C}$ be a finitely complete category and $R \overset{d_0}{\underset{d_1}{\rightrightarrows}} X$ a relation in $\mathcal{C}$. Let $P$ be the pullback of $d_0$ along $d_1$:

$$\begin{tikzcd}
P \arrow[r, "p_0"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & R \arrow[d, "d_0"] \\
R \arrow[r, "d_1"'] & X
\end{tikzcd}$$

$R$ is a **transitive** relation if there is a morphism $P \xrightarrow{p_2} R$ such that $d_0 \circ p_1 = d_0 \circ p_2$ and $d_1 \circ p_2 = d_1 \circ p_0$.

**Example 2.8.** This time, to prove the equivalence of the definitions of transitive relation in $\mathcal{C} = \text{Set}$, we can prove that $P = \{(r_0, r_1) \in R^2 \mid d_0(r_0) = d_1(r_1)\}$. Therefore, such a $p_2$ exists if and only if for all pairs of couples $(d_0(r_1), d_1(r_1)), (d_0(r_0), d_1(r_0))$ in the relation with $d_1(r_1) = d_0(r_0)$, there is a element $p_2(r_0, r_1) \in R$ such that $(d_0(r_1), d_1(r_0)) = (d_0(p_2(r_0, r_1)), d_1(p_2(r_0, r_1)))$ is in the relation, i.e. if and only if $R$ is transitive in the usual way.

**Definition 2.9.** In a finitely complete category, a **equivalence relation** is a reflexive, symmetric and transitive relation.

**Example 2.10.** In Set, the notion of equivalence relation is the same as the usual one.

As the following lemma says, we already know a lot of equivalence relations in any finitely complete category.

**Lemma 2.11.** In a finitely complete category, every kernel pair is a equivalence relation.

*Proof.* Suppose that

$$\begin{tikzcd}
R \arrow[r, "d_0"] \arrow[d, "d_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

is a pullback. First, we proof that $R$ is a relation: Suppose there are two arrows $Z \overset{g}{\underset{h}{\rightrightarrows}} R$ such that $(d_0, d_1) \circ g = (d_0, d_1) \circ h$. So the diagram

$$\begin{tikzcd}
Z \arrow[drr, bend left, "d_0\circ g"] \arrow[ddr, bend right, "d_1\circ g"'] & & \\
& R \arrow[r, "d_0"] \arrow[d, "d_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\
& X \arrow[r, "f"'] & Y
\end{tikzcd}$$

commutes. Hence, by the universal property of pullbacks, $g = h$ and $(d_0, d_1)$ is a monomorphism. Now, let's proof that $R$ is an equivalence: Since

$$\begin{tikzcd}
X \arrow[rrd, bend left, "1_X"] \arrow[rdd, bend right, "1_X"'] & & \\
& R \arrow[r, "d_0"] \arrow[d, "d_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\
& X \arrow[r, "f"'] & Y
\end{tikzcd}$$

commutes, there is an arrow $X \xrightarrow{k} R$ such that $d_1 \circ k = d_0 \circ k = 1_X$. Therefore, $(1_X, 1_X)$ factorizes through $(d_0, d_1)$ and $R$ is reflexive. Moreover, since

$$\begin{tikzcd}
R \arrow[rrd, bend left, "d_1"] \arrow[rdd, bend right, "d_0"'] & & \\
& R \arrow[r, "d_0"] \arrow[d, "d_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\
& X \arrow[r, "f"'] & Y
\end{tikzcd}$$

commute, there exists $R \xrightarrow{\sigma} R$ such that $d_0 \circ \sigma = d_1$ and $d_1 \circ \sigma = d_0$. So, $R$ is symmetric. Finally, if $P$ is the pullback of $d_0$ along $d_1$:

$$\begin{tikzcd}
P \arrow[r, "p_0"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & R \arrow[d, "d_0"] \\
R \arrow[r, "d_1"'] & X
\end{tikzcd}$$

the diagram

$$\begin{tikzcd}
P \arrow[rrd, bend left, "d_0 \circ p_1"] \arrow[rdd, bend right, "d_1 \circ p_0"'] & & \\
& R \arrow[r, "d_0"] \arrow[d, "d_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\
& X \arrow[r, "f"'] & Y
\end{tikzcd}$$

commutes. Thus, there is a morphism $P \xrightarrow{p_2} R$ such that $d_0 \circ p_1 = d_0 \circ p_2$ and $d_1 \circ p_2 = d_1 \circ p_0$. So $R$ is transitive.

□

This lemma leads us naturally to the following definition.

**Definition 2.12.** In a finitely complete category, a equivalence relation is said to be **effective** if it is a kernel pair.

## 2.2 Definition and examples of semi-abelian categories

We are now able to define a semi-abelian category.

**Definition 2.13.** A **regular** category is a finitely complete category where every kernel pair has a coequalizer and where pullbacks preserve regular epimorphisms.

**Definition 2.14.** A category is **Barr exact** if it is regular and if every equivalence relation is effective.

**Definition 2.15.** A pointed category with kernels is called **Bourn protomodular** if it satisfies the regular Short Five Lemma, i.e., if for all commutative diagrams

$$\begin{tikzcd}
\mathrm{Ker}\ f \arrow[r, "\ker f"] \arrow[d, "k"'] & A \arrow[r, "f"] \arrow[d, "a"] & B \arrow[d, "b"] \\
\mathrm{Ker}\ f' \arrow[r, "\ker f'"'] & A' \arrow[r, "f'"'] & B'
\end{tikzcd}$$

where $f$ and $f'$ are regular epimorphisms and $k$ and $b$ are isomorphisms, we have that $a$ is also an isomorphism.

**Definition 2.16.** A category $\mathcal{A}$ is **semi-abelian** when it is pointed, Barr exact, Bourn protomodular and has binary coproducts.

Fortunately, an abelian category is semi-abelian (see example 2.18). But there are other examples. The most frequent example of a semi-abelian category which is not abelian is Gp.

**Example 2.17.** The category Gp of groups is semi-abelian. Indeed, we already know that it is pointed, complete and cocomplete. Moreover, we know that every epimorphism is normal and that the pullback of $G \xrightarrow{f} K$ along $H \xrightarrow{f'} K$ is

$$P = \{(g,h) \in G \times H \mid f(g) = f'(h)\}$$

with the canonical projections. So, if $f$ is an epimorphism, $P \xrightarrow{\pi_2} H$ is also a (regular) epimorphism. In addition, we can prove as we did for Set that an equivalence relation in Gp is an equivalence relation in the usual sense which is compatible with the group law. So, for each equivalence relation $R \leqslant G \times G$, the congruence class of 1, [1], is a normal subgroup of $G$ and $R$ is the kernel pair of $G \xrightarrow{\pi} G/_{[1]}$. It remains to prove that Gp is Bourn protomodular. To do so, it is enough to prove that $a$ is injective and surjective which is straightforward since we can do it elementwise.

**Example 2.18.** Every abelian category is semi-abelian. Indeed, we already know that such a category is pointed, finitely complete and cocomplete, Bourn protomodular and pullbacks preserve (regular) epimorphisms. So it remains to show that every equivalence relation is effective.

Let $R \underset{d_1}{\overset{d_0}{\rightrightarrows}} X$ be an equivalence relation in an abelian category $\mathcal{A}$. Since this relation is reflexive, there is a map $X \xrightarrow{s} R$ such that $d_0 s = d_1 s = 1_X$. Let us write $k = \ker d_1 \in \mathcal{A}(K,R)$ and $n = d_0 k \in \mathcal{A}(K,X)$. If there is a map $x$ such that $nx = 0$, we deduce that $d_0 kx = 0$ and $d_1 kx = 0$. But $(d_0, d_1)$ is a monomorphism, so $kx = 0$ and $x = 0$. This implies that $n$ is a monomorphism. Let $q = \operatorname{coker} n \in \mathcal{A}(X,Y)$. Thus $n = \ker q$. Moreover, we have that $qd_1k = 0 = qn = qd_0k$. But $d_1(1_X - sd_1) = 0$, so $1_X - sd_1$ factors through $k$. Therefore $qd_1(1_X - sd_1) = qd_0(1_X - sd_1)$ which implies $qd_1 = qd_0$. Consider the kernel pair of $q$.

$$\begin{tikzcd}
R \arrow[rd, "t"] \arrow[rrd, bend left, "d_1"] \arrow[rdd, bend right, "d_0"'] & & \\
& S \arrow[r, "s_1"] \arrow[d, "s_2"'] \arrow[rd, phantom, "\lrcorner", very near start] & X \arrow[d, "q"] \\
& X \arrow[r, "q"'] & Y
\end{tikzcd}$$

We know there is a unique map $R \xrightarrow{t} S$ such that $s_1t = d_1$ and $s_2t = d_0$. Since $q$ is an epimorphism, $s_1$ and $s_2$ are also epimorphisms. By definition of $t$ and $k$, $s_1tk = d_1k = 0$. Let's prove that $tk = \ker s_1$. Consider a map $Z \xrightarrow{z} S$ such that $s_1z = 0$. Since $qs_2z = qs_1z = 0$, there exists a morphism $Z \xrightarrow{m} K$ such that $s_2z = nv = d_0kv = s_2tkv$. But $s_1z = 0 = s_1tkv$. Consequently, by definition of $s_1$ and $s_2$, $z = tkv$. In addition, $n = d_0k = s_2tk$ is a monomorphism, so such a $v$ is unique. Thus $tk = \ker s_1$ and

$$\begin{tikzcd}
K \arrow[r, "k"] \arrow[d, equal] & R \arrow[r, "d_1"] \arrow[d, "t"] & X \arrow[d, equal] \\
K \arrow[r, "\ker s_1"'] & S \arrow[r, "s_1"'] & X
\end{tikzcd}$$

commutes. Notice that since $d_1$ and $s_1$ are epimorphisms, the rows are exact. Thus we can apply the Short Five Lemma and deduce that $t$ is an isomorphism. Therefore, $R \overset{d_0}{\underset{d_1}{\rightrightarrows}} X$ is the kernel pair of $q$ and the equivalence relation is effective.

Notice that we did not use the fact that the relation was symmetric and transitive. Actually, in every semi-abelian category, reflexive relations are transitive and symmetric (see section 2.4).

## 2.3 Image factorisation

In order to define exact sequences in semi-abelian categories, we need a image factorisation. To do so, we only need to work with regular categories.

**Lemma 2.19.** Let $\mathcal{A}$ be a category with kernel pairs and their coequalizer. If $Q \overset{q_1}{\underset{q_2}{\rightrightarrows}} A$ is the kernel pair of $f \in \mathcal{A}(A, B)$ and if $A \xrightarrow{p} I$ is the coequalizer of $q_1$ and $q_2$, then $Q \overset{q_1}{\underset{q_2}{\rightrightarrows}} A$ is the kernel pair of $p$.

*Proof.* We know that $pq_1 = pq_2$. Since $fq_1 = fq_2$, there is a morphism $I \xrightarrow{i} B$ such that $ip = f$.

$$\begin{tikzcd}
Q \arrow[r, shift left, "q_1"] \arrow[r, shift right, "q_2"'] & A \arrow[r, "p"] \arrow[rd, "f"'] & I \arrow[d, "i"] \\
& & B
\end{tikzcd}$$

Let $Z \underset{y}{\overset{x}{\rightrightarrows}} A$ be two morphisms such that $px = py$. Thus, $fx = ipx = ipy = fy$. Since $Q \underset{q_2}{\overset{q_1}{\rightrightarrows}} A$ is the kernel pair of $f$ , there is an unique morphism $Z \xrightarrow{m} Q$ such that $q_1 m = x$ and $q_2 m = y$. So $Q \underset{q_2}{\overset{q_1}{\rightrightarrows}} A$ is the kernel pair of $p$.

□

Here is the expected image factorisation.

**Proposition 2.20** (Image Factorisation)**.** Let $\mathcal{A}$ be a regular category. Every morphism $f \in \mathcal{A}(A, B)$ can be written as $f = ip$, where $i$ is a monomorphism and $p$ a regular epimorphism. Moreover, this factorisation is unique up to isomorphism.

*Proof.* Let $Q \underset{q_2}{\overset{q_1}{\rightrightarrows}} A$ be the kernel pair of $f$ and $A \xrightarrow{p} I$ the coequalizer of $q_1$ and $q_2$. By definition, $p$ is a regular epimorphism. Since $fq_1 = fq_2$, there is a unique map $I \xrightarrow{i} B$ such that $f = ip$. So, it remains to show that $i$ is a monomorphism.

$$\begin{tikzcd}
Q \arrow[r, shift left, "q_1"] \arrow[r, shift right, "q_2"'] & A \arrow[r, "p"] \arrow[dr, "f"'] & I \arrow[d, "i"] \\
 & & B
\end{tikzcd}$$

Let $R \underset{r_2}{\overset{r_1}{\rightrightarrows}} I$ be the kernel pair of $i$ and $I \xrightarrow{k} K$ the coequalizer of $r_1$ and $r_2$. Let's prove that $Q \underset{q_2}{\overset{q_1}{\rightrightarrows}} A$ is the kernel pair of $kp$: Firstly, we know that $kpq_1 = kpq_2$. Now, suppose there are two maps $Z \underset{y}{\overset{x}{\rightrightarrows}} A$ such that $kpx = kpy$.

$$\begin{tikzcd}
Z \arrow[drr, bend left, "x"] \arrow[ddr, bend right, "y"'] & & \\
 & Q \arrow[r, "q_1"] \arrow[d, "q_2"'] & A \arrow[d, "kp"] \\
 & A \arrow[r, "kp"'] & K
\end{tikzcd}$$

But, by lemma 2.19, $R \underset{r_2}{\overset{r_1}{\rightrightarrows}} I$ is the kernel pair of $k$. So there is a morphism $Z \xrightarrow{z} R$ such that $r_1 z = px$ and $r_2 z = py$.

$$\begin{tikzcd}
Z \arrow[dr, "z"] \arrow[drr, bend left, "px"] \arrow[ddr, bend right, "py"'] & & \\
 & R \arrow[r, "r_1"] \arrow[d, "r_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & I \arrow[d, "k"] \\
 & I \arrow[r, "k"'] & K
\end{tikzcd}$$

Hence, $fx = ipx = ir_1z = ir_2z = ipy = fy$. Therefore, there is a unique map $Z \xrightarrow{m} Q$ such that $q_1m = x$ and $q_2m = y$.

$$\begin{tikzcd}
Z \arrow[dr, "m"] \arrow[drr, bend left, "x"] \arrow[ddr, bend right, "y"'] & & \\
& Q \arrow[r, "q_1"] \arrow[d, "q_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\
& A \arrow[r, "f"'] & B
\end{tikzcd}$$

So we have just proved that $Q \overset{q_1}{\underset{q_2}{\rightrightarrows}} A$ is the kernel pair of $kp$. But we know that $pq_1 = pq_2$. This implies that there is a morphism $K \xrightarrow{n} I$ such that $nkp = p$. Since $p$ is a (regular) epimorphism, we deduce that $nk = 1_I$ and thus $k$ is a monomorphism. Since $\mathcal{A}$ is a regular category, pullbacks preserve monomorphisms and regular epimorphisms. Recall that $R \overset{r_1}{\underset{r_2}{\rightrightarrows}} I$ is the kernel pair of $k$. Therefore, $r_1$ and $r_2$ are both monomorphisms and regular epimorphisms. So, they are both isomorphisms. But by definition $R \overset{r_1}{\underset{r_2}{\rightrightarrows}} I$ is the kernel pair of $i$. So this can happen only if $i$ is a monomorphism. Indeed, there is a morphism $I \xrightarrow{r} R$ such that $r_1r = 1_I = r_2r$. Thus $r_1 = r_2$ since they are both isomorphisms. Therefore, if there are two maps $C \overset{c_1}{\underset{c_2}{\rightrightarrows}} I$ such that $ic_1 = ic_2$, there is a map $C \xrightarrow{c} R$ with $c_1 = r_1c = r_2c = c_2$.

For uniqueness, suppose there are morphisms $A \xrightarrow{p'} I' \xrightarrow{i'} B$ such that $f = i'p'$, $i'$ is a monomorphism and $p'$ a regular epimorphism. We know that $i'p'q_1 = fq_1 = fq_2 = i'p'q_2$, and so $p'q_1 = p'q_2$. But $p$ is the coequalizer of $Q \overset{q_1}{\underset{q_2}{\rightrightarrows}} A$ . Thus there is a map $I \xrightarrow{j} I'$ such that $jp = p'$.

$$\begin{tikzcd}
Q \arrow[r, shift left, "q_1"] \arrow[r, shift right, "q_2"'] & A \arrow[r, "p"] \arrow[dr, "p'"'] & I \arrow[d, "j"] \\
& & I'
\end{tikzcd}$$

Since $p'$ is a regular epimorphism, we can find two arrows $X \overset{x_1}{\underset{x_2}{\rightrightarrows}} A$ which have $p'$ as coequalizer. So we have $ipx_1 = i'p'x_1 = i'p'x_2 = ipx_2$ and $px_1 = px_2$ since $i$ is a monomorphism. Therefore, there is a map $I' \xrightarrow{j'} I$ such that $j'p' = p$.

$$\begin{tikzcd}
X \arrow[r, shift left, "x_1"] \arrow[r, shift right, "x_2"'] & A \arrow[r, "p'"] \arrow[dr, "p"'] & I' \arrow[d, "j'"] \\
& & I
\end{tikzcd}$$

Thus $jj'p' = jp = p'$ and $jj' = 1_{I'}$ since $p'$ is an epimorphism. We can prove in a similar way that $j'j = 1_I$, hence $j$ and $j'$ are isomorphisms. Finally, $ip = i'jp$. Thus $i = i'j$ and the factorisation is unique up to isomorphism.

□

This proposition leads us to the following definition.

**Definition 2.21.** Let $A \xrightarrow{f} B$ be a morphism in a regular category $\mathcal{A}$. If $f = ip$ with $i \in \mathcal{A}(I, B)$ a monomorphism and $p \in \mathcal{A}(A, I)$ a regular epimorphism, we know that $p$ is the coequalizer of the kernel pair of $f$. We call $i$ and $p$ respectively the **image** and the **coimage** of $f$ and we write $i = \operatorname{im} f$, $p = \operatorname{coim} f$ and $I = \operatorname{Im} f$. This factorisation is called the **image factorisation** of $f$.

As in the abelian case, we can prove the following property of the image factorisation.

**Proposition 2.22.** In a regular category $\mathcal{A}$, the image factorisation is functorial, i.e. if we have a commutative diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[d, "b"] \\
A' \arrow[r, "f'"'] & B'
\end{tikzcd}$$

with $A \xrightarrow{p} I \xrightarrow{i} B$ and $A' \xrightarrow{p'} I' \xrightarrow{i'} B'$ the image factorisations of $f$ and $f'$ respectively, then there is a unique morphism $I \xrightarrow{j} I'$ such that

$$\begin{tikzcd}
A \arrow[r, "p"] \arrow[d, "a"'] & I \arrow[r, "i"] \arrow[d, "j"] & B \arrow[d, "b"] \\
A' \arrow[r, "p'"'] & I' \arrow[r, "i'"'] & B'
\end{tikzcd}$$

commute.

*Proof.* Let $Q \underset{q_2}{\overset{q_1}{\rightrightarrows}} A$ and $Q' \underset{q_2'}{\overset{q_1'}{\rightrightarrows}} A'$ be the kernel pairs of $f$ and $f'$ respectively. We know that their coequalizers are $p$ and $p'$ respectively. Since $f'aq_1 = bfq_1 = bfq_2 = f'aq_2$, there is a morphism $Q \xrightarrow{g} Q'$ such that $q_1'g = aq_1$ and $q_2'g = aq_2$.

$$\begin{tikzcd}
Q \arrow[dr, "g"] \arrow[drr, bend left, "aq_1"] \arrow[ddr, bend right, "aq_2"'] & & \\
& Q' \arrow[r, "q_1'"] \arrow[d, "q_2'"'] \arrow[dr, phantom, "\lrcorner", very near start] & A' \arrow[d, "f'"] \\
& A' \arrow[r, "f'"'] & B'
\end{tikzcd}$$

Therefore, we can compute $p'aq_1 = p'q_1'g = p'q_2'g = p'aq_2$. Thus, since $p$ is the coequalizer of $Q \underset{q_2}{\overset{q_1}{\rightrightarrows}} A$, there is a unique morphism $I \xrightarrow{j} I'$ such that $jp = p'a$. Finally, to prove that $i'j = bi$, it is enough to show that $i'jp = bip$ since $p$ is an epimorphism. But $i'jp = i'p'a = f'a = bf = bip$ which concludes the proof.

□

Recall that in an arbitrary category, if $gf$ is a regular epimorphism and $f$ is an epimorphism, it follows that $g$ is a regular epimorphism. As corollary of the last proposition, in a regular category, we do not need the assumption that $f$ is an epimorphism.

**Corollary 2.23.** If $gf$ is a regular epimorphism in a regular category, then $g$ is a regular epimorphism.

*Proof.* Let $g = ip$ the image factorisation of $g$. Of course, the image factorisation of $gf$ is $1 \circ (gf)$. So, by proposition 2.22, there exists a morphism $j$ making the following diagram commute.

$$\begin{tikzcd}
A \arrow[r, "gf"] \arrow[d, "f"] & C \arrow[r, equal] \arrow[d, "j"] & C \arrow[d, equal] \\
B \arrow[r, "p"'] & I \arrow[r, "i"'] & C
\end{tikzcd}$$

Thus $ij = 1_C$, so $i$ is a split epimorphism and a monomorphism. Hence $i$ is an isomorphism and $g$ a regular epimorphism.

□

## 2.4 Finite cocompleteness

Another interesting property that a category may have is completeness or cocompleteness. We already know that semi-abelian categories are finitely complete. Let's prove now that they are finitely cocomplete. First we need a proposition saying that semi-abelian categories are Mal'cev.

**Definition 2.24.** A finitely complete category is said to be **Mal'cev** if each reflexive relation is an equivalence relation.

**Lemma 2.25.** If in a finitely complete category $\mathcal{C}$, each reflexive relation is transitive, then $\mathcal{C}$ is Mal'cev.

*Proof.* Let $R \underset{d_1}{\overset{d_0}{\rightrightarrows}} X$ be a reflexive relation in $\mathcal{C}$. We have to prove that it is symmetric. We know that $R \xrightarrow{r=(d_0,d_1)} X \times X$ is a monomorphism and there is a map $X \xrightarrow{s_0} R$ such that $rs_0 = (1_X, 1_X)$, i.e. $d_0 s_0 = d_1 s_0 = 1_X$. Let $R \times R \underset{\pi_1}{\overset{\pi_0}{\rightrightarrows}} R$ and $X \times X \underset{\pi'_1}{\overset{\pi'_0}{\rightrightarrows}} X$ be the respective projections. Denote by $R \times R \xrightarrow{d_0 \times d_1} X \times X$ the morphism such that $\pi'_i \circ (d_0 \times d_1) = d_i \pi_i$ for $i \in \{1, 2\}$. Consider the pullback square

$$\begin{tikzcd}
S \arrow[r, "s"] \arrow[d, "\sigma"'] \arrow[dr, phantom, "\lrcorner", very near start] & R \arrow[d, "r"] \\
R \times R \arrow[r, "d_0 \times d_1"'] & X \times X
\end{tikzcd}$$

where $\sigma$ is a monomorphism as $r$ is. Thus $S \underset{\pi_1 \sigma}{\overset{\pi_0 \sigma}{\rightrightarrows}} R$ is a relation. Moreover, we have the identities $d_i s = d_i \pi_i \sigma$ for $i \in \{1, 2\}$. Now, since $\pi'_i \circ (d_0 \times d_1) \circ (1_R, 1_R) = d_i = \pi'_i r$ for

$i \in \{1,2\}$, there is a map $R \xrightarrow{m} S$ making the following diagram commute.

$$\begin{tikzcd}
R \arrow[dr, "m"] \arrow[drr, bend left, "1_R"] \arrow[ddr, bend right, "{(1_R,1_R)}"'] & & \\
& S \arrow[r, "s"] \arrow[d, "\sigma"'] \arrow[dr, phantom, "\lrcorner", very near start] & R \arrow[d, "r"] \\
& R \times R \arrow[r, "d_0 \times d_1"'] & X \times X
\end{tikzcd}$$

Therefore, the relation $S \underset{\pi_1\sigma}{\overset{\pi_0\sigma}{\rightrightarrows}} R$ is reflexive, and so transitive by assumption. Consider now the pullback

$$\begin{tikzcd}
Q \arrow[r, "q_0"] \arrow[d, "q_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & R \arrow[d, "\pi_0\sigma"] \\
S \arrow[r, "\pi_1\sigma"'] & R
\end{tikzcd}$$

By transitivity of the relation $S$, we have a morphism $Q \xrightarrow{q_2} S$ such that $\pi_0\sigma q_1 = \pi_0\sigma q_2$ and $\pi_1\sigma q_2 = \pi_1\sigma q_0$. Because $\pi'_i r s_0 d_1 = d_1 = \pi'_i \circ (d_0 \times d_1) \circ (s_0 d_1, 1_R)$ for $i \in \{1,2\}$, there exists a morphism $R \xrightarrow{v} S$ such that

$$\begin{tikzcd}
R \arrow[dr, "v"] \arrow[drr, bend left, "s_0 d_1"] \arrow[ddr, bend right, "{(s_0 d_1,1_R)}"'] & & \\
& S \arrow[r, "s"] \arrow[d, "\sigma"'] \arrow[dr, phantom, "\lrcorner", very near start] & R \arrow[d, "r"] \\
& R \times R \arrow[r, "d_0 \times d_1"'] & X \times X
\end{tikzcd}$$

commutes. Similarly, we have a map $R \xrightarrow{w} S$ such that $sw = s_0 d_0$ and $\sigma w = (1_R, s_0 d_0)$. Thus, by construction, $\pi_0\sigma w = 1_R = \pi_1\sigma v$. So we have a morphism $R \xrightarrow{y} Q$ making

$$\begin{tikzcd}
R \arrow[dr, "y"] \arrow[drr, bend left, "w"] \arrow[ddr, bend right, "v"'] & & \\
& Q \arrow[r, "q_0"] \arrow[d, "q_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & S \arrow[d, "\pi_0\sigma"] \\
& S \arrow[r, "\pi_1\sigma"'] & R
\end{tikzcd}$$

commute. Consequently, we have constructed a morphism $R \xrightarrow{sq_2 y} R$ satisfying

$$\begin{aligned}
d_0 s q_2 y &= d_0 \pi_0 \sigma q_2 y \\
&= d_0 \pi_0 \sigma q_1 y \\
&= d_0 \pi_0 \sigma v \\
&= d_0 s_0 d_1 \\
&= d_1
\end{aligned}$$

and

$$\begin{aligned} d_1 s q_2 y &= d_1 \pi_1 \sigma q_2 y \\ &= d_1 \pi_1 \sigma q_0 y \\ &= d_1 \pi_1 \sigma w \\ &= d_1 s_0 d_0 \\ &= d_0 \end{aligned}$$

which proves the symmetry of $R$.

□

With this lemma, we can now prove that semi-abelian categories are Mal'cev.

**Proposition 2.26.** Every semi-abelian category is Mal'cev.

*Proof.* Let $R \overset{d_0}{\underset{d_1}{\rightrightarrows}} X$ be a reflexive relation in $\mathcal{A}$, a semi-abelian category. By lemma 2.25, it is enough to prove that $R$ is transitive. We know that $R \xrightarrow{r=(d_0,d_1)} X \times X$ is a monomorphism and there is a map $X \xrightarrow{s_0} R$ such that $rs_0 = (1_X, 1_X)$, i.e. $d_0 s_0 = d_1 s_0 = 1_X$. Let $K$ be the limit of the following triangle diagram.

$$\begin{tikzcd}
& & X & & \\
& R \arrow[ur, "d_1"] \arrow[ddl, "d_0"'] & & R \arrow[ul, "d_0"'] \arrow[ddr, "d_1"] & \\
& & K \arrow[uu, "d'_2"] \arrow[ul, "p'_0"] \arrow[ur, "p'_2"'] \arrow[dll, "d'_0"] \arrow[drr, "d'_1"] \arrow[d, "p'_1"] & & \\
X & & R \arrow[ll, "d_0"] \arrow[rr, "d_1"'] & & X
\end{tikzcd}$$

Because the following diagram commute, there is a unique morphism $R \xrightarrow{s'_0} K$ such that $d'_0 s'_0 = d_0$, $d'_1 s'_0 = d_1$, $d'_2 s'_0 = d_1$, $p'_0 s'_0 = 1_R$, $p'_1 s'_0 = 1_R$ and $p'_2 s'_0 = s_0 d_1$.

$$\begin{tikzcd}
& & X & & \\
& R \arrow[ur, "d_1"] \arrow[ddl, "d_0"'] & & R \arrow[ul, "d_0"'] \arrow[ddr, "d_1"] & \\
& & R \arrow[uu, "d_1"] \arrow[ul, equal] \arrow[ur, "s_0 d_1"'] \arrow[dll, "d_0"] \arrow[drr, "d_1"] \arrow[d, equal] & & \\
X & & R \arrow[ll, "d_0"] \arrow[rr, "d_1"'] & & X
\end{tikzcd}$$

Similarly, there is a unique arrow $R \xrightarrow{s_1'} K$ such that $d_0's_1' = d_0$, $d_1's_1' = d_1$, $d_2's_1' = d_0$, $p_0's_1' = s_0d_0$, $p_1's_1' = 1_R$ and $p_2's_1' = 1_R$. Now, consider the pullback $P$ of $d_0$ and $d_1$.

$$\begin{tikzcd}
P \arrow[r, "p_1"] \arrow[d, "p_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & R \arrow[d, "d_1"] \\
R \arrow[r, "d_0"'] & X
\end{tikzcd}$$

Since $d_1p_0' = d_0p_2'$, there is a morphism $K \xrightarrow{g} P$ such that $p_1g = p_0'$ and $p_0g = p_2'$. Let $a$ and $b$ be the unique morphisms making the following diagram commute.

$$\begin{tikzcd}
\operatorname{Ker} d_1 \arrow[r, "a"] \arrow[d, "\ker d_1"'] & \operatorname{Ker} d_2' \arrow[r, "b"] \arrow[d, "\ker d_2'"] & \operatorname{Ker} d_0 \arrow[d, "\ker d_0"] \\
R \arrow[r, "s_0'"'] \arrow[d, "d_1"'] & K \arrow[r, "p_2'"'] \arrow[d, "d_2'"] & R \arrow[d, "d_0"] \\
X \arrow[r, equal] & X \arrow[r, equal] & X
\end{tikzcd}$$

We know that $ba = 0$ since $(\ker d_0)ba = p_2's_0' \ker d_1 = s_0d_1 \ker d_1 = 0$. Let's prove that $a = \ker b$. Suppose there is a map $Z \xrightarrow{z} \operatorname{Ker} d_2'$ such that $bz = 0$. Thus, $p_2'(\ker d_2')z = (\ker d_0)bz = 0$. So, $d_1p_1'(\ker d_2')z = d_1p_2'(\ker d_2')z = 0$. Therefore, there is a map $Z \xrightarrow{m} \operatorname{Ker} d_1$ such that $(\ker d_1)m = p_1'(\ker d_2')z$. Thus, $(\ker d_2')am = s_0'p_1'(\ker d_2')z$ and $p_0'(\ker d_2')am = p_1'(\ker d_2')z$. So, we can compute

$$\begin{aligned}
d_0p_0'(\ker d_2')am &= d_0p_1'(\ker d_2')z \\
&= d_0p_0'(\ker d_2')z
\end{aligned}$$

and

$$\begin{aligned}
d_1p_0'(\ker d_2')am &= d_2'(\ker d_2')am \\
&= 0 \\
&= d_1p_0'(\ker d_2')z.
\end{aligned}$$

Therefore, since $r = (d_0, d_1)$ is a monomorphism, $p_0'(\ker d_2')am = p_0'(\ker d_2')z$. Moreover,

$$\begin{aligned}
p_2'(\ker d_2')am &= (\ker d_0)bam \\
&= 0 \\
&= p_2'(\ker d_2')z
\end{aligned}$$

and so $p_1'(\ker d_2')am = p_1'(\ker d_2')z$ again by the fact that $r$ is a monomorphism and $d_0p_1' = d_0p_0'$ and $d_1p_1' = d_1p_2'$. So, by the universal property of $K$, we deduce that $(\ker d_2')am = (\ker d_2')z$ and $am = z$. In addition, $(\ker d_2')a = s_0' \ker d_1$ and $s_0'$ is a (split) monomorphism. So $a$ is also a monomorphism and such a $m$ is unique. Therefore $a = \ker b$.

Now let's prove that $b$ is a regular epimorphism. Notice that $d_2's_1' \ker d_0 = d_0 \ker d_0 = 0$. So there is a map $\operatorname{Ker} d_0 \xrightarrow{k} \operatorname{Ker} d_2'$ such that $(\ker d_2')k = s_1' \ker d_0$. Thus

$$\begin{aligned}
(\ker d_0)bk &= p_2'(\ker d_2')k \\
&= p_2's_1' \ker d_0 \\
&= \ker d_0
\end{aligned}$$

and so $bk = 1_{\operatorname{Ker} d_0}$. Therefore $b$ is a split epimorphism and thus a regular epimorphism.

Let $c$ and $e$ be the unique morphisms making the following diagram commute (we know that $d_0p_0gs'_0 = d_0p'_2s'_0 = d_0s_0d_1 = d_1$).

$$\begin{tikzcd}
\operatorname{Ker} d_1 \arrow[r, "c"] \arrow[d, "\ker d_1"'] & \operatorname{Ker}(d_0p_0) \arrow[r, "e"] \arrow[d, "\ker(d_0p_0)"] & \operatorname{Ker} d_0 \arrow[d, "\ker d_0"] \\
R \arrow[r, "gs'_0"'] \arrow[d, "d_1"'] & P \arrow[r, "p_0"'] \arrow[d, "d_0p_0"] & R \arrow[d, "d_0"] \\
X \arrow[r, equal] & X \arrow[r, equal] & X
\end{tikzcd}$$

Because $(\ker d_0)ec = p_0gs'_0 \ker d_1 = p'_2s'_0 \ker d_1 = s_0d_1 \ker d_1 = 0$, $ec = 0$. Let us prove that $c = \ker e$. Suppose we have a map $Y \xrightarrow{y} \operatorname{Ker}(d_0p_0)$ such that $ey = 0$. Thus,

$$\begin{aligned}
d_1p_1(\ker(d_0p_0))y &= d_0p_0(\ker(d_0p_0))y \\
&= d_0(\ker d_0)ey \\
&= 0.
\end{aligned}$$

So there exists a morphism $Y \xrightarrow{h} \operatorname{Ker} d_1$ such that $(\ker d_1)h = p_1(\ker(d_0p_0))y$. Let's do some computations:

$$\begin{aligned}
p_0(\ker(d_0p_0))ch &= p_0gs'_0(\ker d_1)h \\
&= p'_2s'_0(\ker d_1)h \\
&= s_0d_1(\ker d_1)h \\
&= 0 \\
&= (\ker d_0)ey \\
&= p_0(\ker(d_0p_0))y
\end{aligned}$$

and

$$\begin{aligned}
p_1(\ker(d_0p_0))ch &= p_1gs'_0(\ker d_1)h \\
&= p'_0s'_0(\ker d_1)h \\
&= (\ker d_1)h \\
&= p_1(\ker(d_0p_0))y.
\end{aligned}$$

So, by definition of $P$, $(\ker(d_0p_0))ch = (\ker(d_0p_0))y$ and $ch = y$. But we know that $(\ker(d_0p_0))c = gs'_0 \ker d_1$ where $gs'_0$ is a monomorphism since $p_1gs'_0 = 1_R$. So, $c$ is also a monomorphism, such a $h$ is unique and $c = \ker e$.

Moreover, $d_0p_0gs'_1 \ker d_0 = d_0p'_2s'_1 \ker d_0 = d_0 \ker d_0 = 0$. So we can find a map $\operatorname{Ker} d_0 \xrightarrow{j} \operatorname{Ker}(d_0p_0)$ such that $(\ker(d_0p_0))j = gs'_1 \ker d_0$. Thus

$$\begin{aligned}
(\ker d_0)ej &= p_0(\ker(d_0p_0))j \\
&= p_0gs'_1 \ker d_0 \\
&= p'_2s'_1 \ker d_0 \\
&= \ker d_0
\end{aligned}$$

and so $ej = 1_{\operatorname{Ker} d_0}$. Therefore $e$ is a regular epimorphism.

Let $f$ be the unique morphism making the following diagram commute.

$$\begin{tikzcd}
\operatorname{Ker} d_2' \arrow[r, "f"] \arrow[d, "\ker d_2'"'] & \ker(d_0p_0) \arrow[d, "\ker(d_0p_0)"] \\
K \arrow[r, "g"] \arrow[d, "d_2'"'] & P \arrow[d, "d_0p_0"] \\
X \arrow[r, equal] & X
\end{tikzcd}$$

We can calculate

$$\begin{aligned}
(\ker(d_0p_0))fa &= g(\ker d_2')a \\
&= gs_0' \ker d_1.
\end{aligned}$$

Thus by definition of $c$, $fa = c$. Similarly,

$$\begin{aligned}
(\ker d_0)ef &= p_0(\ker(d_0p_0))f \\
&= p_0 g \ker d_2' \\
&= p_2' \ker d_2'.
\end{aligned}$$

And so by definition of $b$, $ef = b$.

$$\begin{tikzcd}
\operatorname{Ker} d_1 \arrow[r, "a"] \arrow[d, equal] & \operatorname{Ker} d_2' \arrow[r, "b"] \arrow[d, "f"] & \operatorname{Ker} d_0 \arrow[d, equal] \\
\operatorname{Ker} d_1 \arrow[r, "c"] & \operatorname{Ker}(d_0p_0) \arrow[r, "e"] & \operatorname{Ker} d_0
\end{tikzcd}$$

By the regular Short Five Lemma, $f$ is a isomorphism. But $d_2's_1's_0 = 1_X$ and $d_0p_0gs_1's_0 = 1_X$, so $d_2'$ and $d_0p_0$ are regular epimorphisms.

$$\begin{tikzcd}
\operatorname{Ker} d_2' \arrow[r, "\ker d_2'"] \arrow[d, "f"] & K \arrow[r, "d_2'"] \arrow[d, "g"] & X \arrow[d, equal] \\
\operatorname{Ker}(d_0p_0) \arrow[r, "\ker(d_0p_0)"] & P \arrow[r, "d_0p_0"] & X
\end{tikzcd}$$

Thus, again by the regular Short Five Lemma, $g$ is a isomorphism. Now we can define $p_2 = p_1'g^{-1} \in \mathcal{A}(P, R)$. Thus

$$\begin{aligned}
d_0p_2 &= d_0p_1'g^{-1} \\
&= d_0p_0'g^{-1} \\
&= d_0p_1
\end{aligned}$$

and

$$\begin{aligned}
d_1p_2 &= d_1p_1'g^{-1} \\
&= d_1p_2'g^{-1} \\
&= d_1p_0
\end{aligned}$$

which proves the transitivity of $R$.

□

Now we can prove that semi-abelian categories are finitely cocomplete. This prove can be found in [3].

**Proposition 2.27.** Every semi-abelian category is finitely cocomplete.

*Proof.* Let $\mathcal{A}$ be a semi-abelian category. Since it has a initial object and all binary coproducts, $\mathcal{A}$ has all finite coproducts. So, it is enough to show that it has all coequalizers. Let $f, g \in \mathcal{A}(A, B)$ and $A \xrightarrow{(f,g)} B \times B$ the unique map satisfying $\pi_1 \circ (f, g) = f$ and $\pi_2 \circ (f, g) = g$. Let us write $(f, g) = ip$ for the image factorisation of $(f, g)$.

$$\begin{tikzcd}
A \arrow[r, two heads, "p"] \arrow[rr, bend right, "{(f,g)}"'] & I \arrow[r, tail, "i"] & B \times B \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & B
\end{tikzcd}$$

Since $p$ is an epimorphism, $B \xrightarrow{z} Z$ coequalizes $A \overset{f}{\underset{g}{\rightrightarrows}} B$ if and only if it coequlizes $I \overset{\pi_1 \circ i}{\underset{\pi_2 \circ i}{\rightrightarrows}} B$ . So it is enough to prove that $I \overset{\pi_1 \circ i}{\underset{\pi_2 \circ i}{\rightrightarrows}} B$ has a coequalizer. Consider now the map $I + B \xrightarrow{s} B \times B$ such that $s \circ \iota_I = i$ and $s \circ \iota_B = (1_B, 1_B)$. Let us write $s = rt$, its image factorisation.

$$\begin{tikzcd}
I + B \arrow[dr, two heads, "t"] \arrow[ddrr, bend right, "s"'] & & B \arrow[ll, "\iota_B"'] \arrow[dd, "{(1_B,1_B)}"] & \\
& R \arrow[dr, tail, "r"] & & \\
I \arrow[uu, "\iota_I"] \arrow[rr, tail, "i"'] & & B \times B \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & B
\end{tikzcd}$$

Now, since $t$ is an epimorphism, we observe that $B \xrightarrow{z} Z$ coequalizes $R \overset{\pi_1 \circ r}{\underset{\pi_2 \circ r}{\rightrightarrows}} B$ if and only if it coequalizes $I + B \overset{\pi_1 \circ s}{\underset{\pi_2 \circ s}{\rightrightarrows}} B$ . By the universal property of the coproduct, it happens if and only if $z \circ \pi_1 \circ i = z \circ \pi_2 \circ i$ and $z \circ \pi_1 \circ (1_B, 1_B) = z \circ \pi_2 \circ (1_B, 1_B)$. The second condition being trivial, we have just proved that $I \overset{\pi_1 \circ i}{\underset{\pi_2 \circ i}{\rightrightarrows}} B$ has a coequalizer if and only if $R \overset{\pi_1 \circ r}{\underset{\pi_2 \circ r}{\rightrightarrows}} B$ has one. But this is a relation on $B$. Indeed, the morphism $R \xrightarrow{(\pi_1 \circ r, \pi_2 \circ r)} B \times B$ is nothing but $r$ which is a monomorphism. Moreover, since $(1_B, 1_B) = r \circ t \circ \iota_B$, this is a reflexive relation. But because $\mathcal{A}$ is Mal'cev (see proposition 2.26), $R \overset{\pi_1 \circ r}{\underset{\pi_2 \circ r}{\rightrightarrows}} B$ is an equivalence relation. So, keeping in mind that $\mathcal{A}$ is Barr exact, this is a kernel pair and so it has a coequalizer since $\mathcal{A}$ is regular.

□

## 2.5 Equivalences of epimorphisms

In this section, we shall show that in a semi-abelian category, the notions of strong, regular and normal epimorphisms are equivalent. Let's start by recalling the definition of strong epimorphism.

**Definition 2.28.** In a category $\mathcal{C}$, an epimorphism $f \in \mathcal{A}(A, B)$ is a **strong epimorphism** if for each commutative diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\
C \arrow[r, tail, "m"'] & D
\end{tikzcd}$$

where $m$ is a monomorphism, there exists a (necessarily unique) map $B \xrightarrow{t} C$ such that $tf = g$ and $mt = h$.

Now we prove the equivalence between strong and regular epimorphisms. Notice that we only need to suppose that the category is regular. In particular, this equivalence works in semi-abelian categories.

**Proposition 2.29.** Let $A \xrightarrow{f} B$ be a morphism in a regular category $\mathcal{A}$. Then $f$ is a strong epimorphism if and only if it is a regular epimorphism.

*Proof.* First, suppose that $f$ is a regular epimorphism and that

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\
C \arrow[r, tail, "m"'] & D
\end{tikzcd}$$

is a commutative diagram with $m$ being a monomorphism. Let $\begin{tikzcd} P \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & A \end{tikzcd}$ be a pair of arrows such that $f$ is their coequalizer. So $mgp_1 = hfp_1 = hfp_2 = mgp_2$ implies that $gp_1 = gp_2$. Thus, there exists a morphism $B \xrightarrow{t} C$ such that $tf = g$. But therefore $mtf = mg = hf$ implies that $mt = h$, which proves that $f$ is a strong epimorphism.

Conversely, suppose that $f$ is a strong epimorphism and let $f = ip$ its image factorisation. By definition of strong epimorphism, there is a map $B \xrightarrow{t} I$ making the diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "p"'] & B \arrow[d, equal] \\
I \arrow[r, tail, "i"] & B
\end{tikzcd}$$

commute. Thus $it = 1_B$. But since $i$ is a monomorphism, $iti = i$ implies $ti = 1_I$. So $i$ is an isomorphism and $f$ a regular epimorphism.

□

This proposition has a corollary which is really useful when we work with image factorisation. We shall not always refer to it when it is used.

**Corollary 2.30.** In a regular category, the composition of two regular epimorphisms is a regular epimorphism.

*Proof.* It is enough to show that if $f \in \mathcal{A}(A, B)$ and $g \in \mathcal{A}(B, C)$ are strong epimorphisms, then so is $gf$. Consider a commutative square

$$\begin{tikzcd}
A \arrow[r, "gf"] \arrow[d, "k"'] & C \arrow[d, "h"] \\
D \arrow[r, tail, "m"'] & E
\end{tikzcd}$$

where $m$ is a monomorphism. Since $f$ is a strong epimorphism, there is a map $B \xrightarrow{t'} D$ making the diagram on the left commute.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "k"'] & B \arrow[d, "hg"] \arrow[ld, "t'"] & B \arrow[r, "g"] \arrow[d, "t'"'] & C \arrow[d, "h"] \arrow[ld, "t"] \\
D \arrow[r, tail, "m"'] & E & D \arrow[r, tail, "m"'] & E
\end{tikzcd}$$

Now, since $g$ is a strong epimorphism, there is a morphism $C \xrightarrow{t} D$ such that the right-hand square commutes. So $mt = h$ and $tgf = t'f = k$ which proves that $gf$ is a strong epimorphism.

□

Now, if we come back to our factorisation, we can prove that in the case where $\mathcal{A}$ is semi-abelian (not only regular), we actually have a (normal epi - mono) factorisation. Indeed, we can prove that every regular epimorphism is normal.

**Proposition 2.31.** In a semi-abelian category, every regular epimorphism is a normal epimorphism.

*Proof.* Let $B \xrightarrow{p} C$ be a regular epimorphism in $\mathcal{A}$, a semi-abelian category. Let $q = \operatorname{coker}(\ker p) \in \mathcal{A}(B, C')$. We have to prove that $p$ and $q$ are isomorphic. Since $\ker p$ is a normal monomorphism, $\ker q = \ker(\operatorname{coker}(\ker p)) = \ker p$. By definition of $q$ and since $p \circ \ker p = 0$, there is a unique morphism $C' \xrightarrow{m} C$ such that $mq = p$. Let $A \overset{x}{\underset{y}{\rightrightarrows}} B$ and $A' \overset{x'}{\underset{y'}{\rightrightarrows}} B$ be the kernel pairs of $p$ and $q$ respectively. Since $px' = mqx' = mqy' = py'$, there is a morphism $A' \xrightarrow{a} A$ such that $xa = x'$ and $ya = y'$.

$$\begin{tikzcd}
A' \arrow[rd, "a"] \arrow[rrd, bend left, "x'"] \arrow[rdd, bend right, "y'"'] & & \\
& A \arrow[r, "x"] \arrow[d, "y"'] \arrow[rd, phantom, "\lrcorner", very near start] & B \arrow[d, "p"] \\
& B \arrow[r, "p"'] & C
\end{tikzcd}$$

Moreover, since $p \circ 0 = 0 = p \circ \ker p$, there is a unique map $\operatorname{Ker} p \xrightarrow{k} A$ such that $xk = 0$ and $yk = \ker p$. Because $\ker p = \ker q$, we have also a unique map $\operatorname{Ker} p \xrightarrow{k'} A'$ such that $x'k' = 0$ and $y'k' = \ker p$.

$$\begin{tikzcd}
\operatorname{Ker} p \arrow[rd, "k'"] \arrow[rrd, bend left, "0"] \arrow[rdd, bend right, "\ker p"'] & & \\
& A' \arrow[r, "x'"] \arrow[d, "y'"'] \arrow[rd, phantom, "\lrcorner", very near start] & B \arrow[d, "q"] \\
& B \arrow[r, "q"'] & C'
\end{tikzcd}$$

But we know that $xak' = x'k' = 0$ and $yak' = y'k' = \ker p$. So $ak' = k$. Let's prove that $k = \ker x$: We know that $xk = 0$. Suppose there is a map $Z \xrightarrow{z} A$ such that $xz = 0$.

$$\begin{tikzcd}
Z \arrow[rrd, bend left, "z"] \arrow[rdd, bend right] & & \\
 & \operatorname{Ker} p \arrow[r, "k"] \arrow[d] & A \arrow[d, "x"] \\
 & 0 \arrow[r] & B
\end{tikzcd}$$

So $pyz = pxz = 0$. Hence, there is a unique map $Z \xrightarrow{w} \operatorname{Ker} p$ such that $(\ker p) \circ w = yz$.

$$\begin{tikzcd}
Z \arrow[rd, "w"] \arrow[rrd, bend left, "yz"] \arrow[rdd, bend right] & & \\
 & \operatorname{Ker} p \arrow[r, "\ker p"'] \arrow[d] \arrow[rd, phantom, "\lrcorner" very near start] & B \arrow[d, "p"] \\
 & 0 \arrow[r] & C
\end{tikzcd}$$

Thus, $ykw = (\ker p) \circ w = yz$. But since $A \overset{x}{\underset{y}{\rightrightarrows}} B$ is the kernel pair of $p$, $z = kw$ (the following diagram commute).

$$\begin{tikzcd}
Z \arrow[rd, "kw"] \arrow[rd, bend right, "z"'] \arrow[rrd, bend left, "xkw=0"] \arrow[rdd, bend right, "ykw"'] & & \\
 & A \arrow[r, "x"] \arrow[d, "y"'] \arrow[rd, phantom, "\lrcorner" very near start] & B \arrow[d, "p"] \\
 & B \arrow[r, "p"'] & C
\end{tikzcd}$$

But if there is another $Z \xrightarrow{w'} \operatorname{Ker} p$ such that $z = kw'$, we would have $yz = ykw' = (\ker p) \circ w'$ and so $w = w'$ by uniqueness of $w$. So $k = \ker x$. By a similar reasoning, we prove that $k' = \ker x'$. Therefore, this diagram commute.

$$\begin{tikzcd}
\operatorname{Ker} p \arrow[r, tail, "k'"] \arrow[d, equal] & A' \arrow[r, two heads, "x'"] \arrow[d, "a"] & B \arrow[d, equal] \\
\operatorname{Ker} p \arrow[r, tail, "k"'] & A \arrow[r, two heads, "x"'] & B
\end{tikzcd}$$

In addition $x$ and $x'$ are regular epimorphisms since they are the pullback of $p$ and $q$ respectively and $\mathcal{A}$ is regular. Therefore, by the regular Short Five Lemma, $a$ is an isomorphism. So, by definition of $a$, $A' \overset{x'}{\underset{y'}{\rightrightarrows}} B$ is the kernel pair of $p$. So we know that $A' \overset{x'}{\underset{y'}{\rightrightarrows}} B$ is the kernel pair of $p$ and $q$ and that $p$ and $q$ are regular epimorphisms. Therefore, by the image factorisation, we conclude that $p$ and $q$ are both the coequalizer of $A' \overset{x'}{\underset{y'}{\rightrightarrows}} B$ and so they are isomorphic.

□

As corollary, we can now prove two lemmas we have seen in the case of abelian categories.

**Lemma 2.32.** In a semi-abelian category, a morphism $A \xrightarrow{f} B$ is a monomorphism if and only if $\operatorname{Ker} f = 0$.

*Proof.* If $f$ is a monomorphism, it suffices to notice that $fg = 0$ if and only if $g = 0$. Conversely, suppose $\operatorname{Ker} f = 0$. Let $f = ip$ be the image factorisation of $f$. Since $\ker f = 0 \longrightarrow A$, we know that $\ker p = 0 \longrightarrow A$. But $p$ is a normal epimorphism. So, $p = \operatorname{coker}(\ker p) = \operatorname{coker}( 0 \longrightarrow A ) = 1_A$ and $f = i$ is a monomorphism.

□

**Lemma 2.33.** In a semi-abelian category, pullbacks reflect monomorphisms.

*Proof.* Consider a pullback square where $m$ is a monomorphism and take the following kernels:

$$\begin{tikzcd}
0 = \operatorname{Ker} m \arrow[r, "\ker m"] \arrow[d, "a"] & P \arrow[r, "m"] \arrow[d, "n"] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "g"] \\
\operatorname{Ker} f \arrow[r, "\ker f"'] & A \arrow[r, "f"'] & C
\end{tikzcd}$$

By a well-known result about kernels and pullbacks (see lemma 4.2.4 in [4]), the induced map $a$ is an isomorphism. So $\operatorname{Ker} f = 0$ and $f$ is a monomorphism.

□

## 2.6 Exact sequences

Thanks to the (normal epi - mono) factorisation, we can define exact sequences and prove their properties in a similar way that one can do for abelian categories. We recall them here.

**Definition 2.34.** In a semi-abelian category $\mathcal{A}$, a **short exact sequence (s.e.s.)** is a sequence of morphisms

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \tag{2.1}$$

such that $f = \ker g$ and $g = \operatorname{coker} f$.

**Definition 2.35.** In a semi-abelian category, a sequence $A \xrightarrow{f} B \xrightarrow{g} C$ is **exact at** $B$ if $\operatorname{im} f = \ker g$.

**Definition 2.36.** In a semi-abelian category, a sequence $\cdots A_{n+1} \longrightarrow A_n \longrightarrow A_{n-1} \cdots$ is **exact** if it is exact at each internal $A_n$.

There is a well-known link between exactness of a sequence and its image factorisation.

**Lemma 2.37.** If $A \xrightarrow{f} B \xrightarrow{g} C$ is a sequence in a semi-abelian category, and if $f = ip$ and $g = jq$ are the image factorisations of $f$ and $g$, then $A \xrightarrow{f} B \xrightarrow{g} C$ is exact if and only if $0 \longrightarrow I \xrightarrow{i} B \xrightarrow{q} J \longrightarrow 0$ is a short exact sequence.

*Proof.* Let us proof the 'if' part:
We know that $i = \ker q$ and $q = \operatorname{coker} i$. So $jqi = 0$. If $Z \xrightarrow{z} B$ is such that $jqz = 0$,

$$\begin{tikzcd}
Z \arrow[drr, bend left, "z"] \arrow[ddr, bend right] & & \\
& I \arrow[r, "i"] \arrow[d] & B \arrow[d, "jq"] \\
& 0 \arrow[r] & C
\end{tikzcd}$$

then $qz = 0$ since $j$ is a monomorphism. Thus, there is a unique map $Z \xrightarrow{m} I$ such that $im = z$ because $i = \ker q$. Therefore, $i = \ker jq = \ker g$ and $A \xrightarrow{f} B \xrightarrow{g} C$ is exact.
For the 'only if' part, we know that $i = \ker jq$. So $qi = 0$ since $jqi = 0$ and $j$ is a monomorphism. Suppose there is a map $X \xrightarrow{x} B$ such that $qx = 0$.

$$\begin{tikzcd}
X \arrow[drr, bend left, "x"] \arrow[ddr, bend right] & & \\
& I \arrow[r, "i"] \arrow[d] & B \arrow[d, "q"] \\
& 0 \arrow[r] & J
\end{tikzcd}$$

So $jqx = 0$. Thus, there is a unique map $X \xrightarrow{n} I$ such that $in = x$. Therefore, $i = \ker q$. But $q$ is a normal epimorphism. So $q = \operatorname{coker}(\ker q) = \operatorname{coker} i$ and we have proved that $0 \longrightarrow I \xrightarrow{i} B \xrightarrow{q} J \longrightarrow 0$ is a short exact sequence.

□

There are many examples of exact sequences.

**Lemma 2.38.** In a semi-abelian category, the following equivalences hold:

1. $0 \longrightarrow A \xrightarrow{f} B$ is exact if and only if $f$ is a monomorphism.
2. $B \xrightarrow{g} C \longrightarrow 0$ is exact if and only if $g$ is a normal epimorphism.
3. $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C$ is exact if and only if $f = \ker g$.
4. $A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is exact if and only if $g = \operatorname{coker} f$ and $\operatorname{im} f$ is a normal monomorphism.
5. $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is exact if and only if it is a short exact sequence.
6. $0 \longrightarrow A \xrightarrow{f} B \longrightarrow 0$ is exact if and only if $f$ is an isomorphism.

*Proof.* Since $0 \longrightarrow A$ is a monomorphism, the first equivalence is exactly the one given given by lemma 2.32. For the second one, notice that $\ker( C \longrightarrow 0 ) = 1_C$ and $g$ is a normal epimorphism if and only if $\operatorname{im} g = 1_C$. The third equivalence follows from the first one.

Let's prove the fourth equivalence: For the 'only if' part, let $f = ip$ be the image factorisation of $f$. We know that $i = \ker g$ and $g$ is a normal epimorphism by the second equivalence. So, $i$ is a normal monomorphism. But since $p$ is an epimorphism, $\operatorname{coker} f = \operatorname{coker} i$. So $g = \operatorname{coker}(\ker g) = \operatorname{coker} i = \operatorname{coker} f$. For the 'if' part, again, let $f = ip$ be the image factorisation of $f$. We know that $\operatorname{coker} f = \operatorname{coker} i$ and $i = \ker(\operatorname{coker} i)$. So, $i = \ker(\operatorname{coker} i) = \ker(\operatorname{coker} f) = \ker g$ and $g$ is a normal epimorphism since $g = \operatorname{coker} f$.

Moreover, the third and fourth equivalences imply the fifth one. (Notice that if $f = \ker g$, then $\operatorname{im} f = f = \ker g$ is a normal monomorphism). Finally, the last equivalence is implied by the first and second one. Indeed, if $f$ is an isomorphism, $f$ is a normal epimorphism since $\operatorname{coker}(\ker f) = \operatorname{coker}(\ 0 \longrightarrow A\ ) = f$.

□

## 2.7 Homology of proper chain complexes

Now, let's define the notion of homology. As for abelian categories, homology 'measures' the exactness of sequences. Unfortunately, to make the theory work, it is not enough to assume that the sequence is a complex. Indeed, we have to make the assumption that morphisms are 'proper'.

**Definition 2.39.** A morphism $f$ in a semi-abelian category is a **proper morphism** if $\operatorname{im} f$ is a normal monomorphism.

**Definition 2.40.** A **complex (or chain complex)** in a semi-abelian category $\mathcal{A}$ is a sequence of morphisms

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "d_{n+1}"] & C_n \arrow[r, "d_n"] & C_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

such that $d_n d_{n+1} = 0$ for all $n \in \mathbb{Z}$. A complex is called a **proper chain complex** if $d_n$ is a proper morphism for all $n \in \mathbb{Z}$. We denote by $\operatorname{Ch}\mathcal{A}$ the category which has chain complexes in $\mathcal{A}$ as objects and a morphism $f \in \operatorname{Ch}\mathcal{A}(C_\bullet, D_\bullet)$ is the data of maps $f_n \in \mathcal{A}(C_n, D_n)$ for all $n \in \mathbb{Z}$ such that

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "f_{n+1}"'] & C_n \arrow[r, "d_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r] & D_{n+1} \arrow[r, "d'_{n+1}"'] & D_n \arrow[r, "d'_n"'] & D_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

commutes. Let $\operatorname{PCh}\mathcal{A}$ be the full subcategory of $\operatorname{Ch}\mathcal{A}$ of proper chain complexes.

We can prove some useful lemmas to identify proper morphisms.

**Lemma 2.41.** Let $\mathcal{A}$ be a semi-abelian category, $f \in \mathcal{A}(A, B)$ and $m \in \mathcal{A}(B, C)$ such that $m$ is a monomorphism and $mf$ is a proper morphism. Then $f$ is also a proper morphism.

*Proof.* Let $f = ip$ be the image factorisation of $f$. So, since $m$ is a monomorphism, $mi = \operatorname{im}(mf)$ and $mi$ is a normal monomorphism. Suppose $mi = \ker g$ where $g \in \mathcal{A}(C, D)$. So $gmi = 0$. If there is a map $Z \xrightarrow{z} B$ such that $gmz = 0$,

$$\begin{tikzcd}
Z \arrow[drr, bend left, "z"] \arrow[ddr, bend right] & & \\
& I \arrow[r, "i"] \arrow[d] & B \arrow[d, "gm"] \\
& 0 \arrow[r] & D
\end{tikzcd}$$

then there is a unique map $Z \xrightarrow{w} I$ such that $miw = mz$, i.e. $iw = z$. So $i = \ker(gm)$ and $f$ is a proper morphism.

□

In order to prove another lemma about proper morphisms, we have to show the following one.

**Lemma 2.42.** Let $\mathcal{A}$ be a semi-abelian category, $f \in \mathcal{A}(A, B)$ and $g \in \mathcal{A}(A, C)$ two regular epimorphisms and the following diagram their pushout.

$$\begin{tikzcd}
A \arrow[r, "g"] \arrow[d, "f"'] & C \arrow[d, "q_1"] \arrow[dl, phantom, "\ulcorner", very near end] \\
B \arrow[r, "q_2"'] & Q
\end{tikzcd}$$

If $P$ is the pullback of $q_1$ and $q_2$ and if $e$ is the unique map making

$$\begin{tikzcd}
A \arrow[dr, "e"] \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "f"'] & & \\
& P \arrow[r, "p_2"] \arrow[d, "p_1"] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "q_1"] \\
& B \arrow[r, "q_2"'] & Q
\end{tikzcd}$$

commute, then $e$ is a regular epimorphism.

*Proof.* Let $F \overset{f_1}{\underset{f_2}{\rightrightarrows}} A$ and $G \overset{g_1}{\underset{g_2}{\rightrightarrows}} A$ be the kernel pairs of $f$ and $g$ respectively. So we know there exist two morphisms $A \xrightarrow{a_1} F$ and $A \xrightarrow{a_2} G$ such that $f_1a_1 = f_2a_1 = g_1a_2 = g_2a_2 = 1_A$. Consider $M$ the pullback of $f_2$ and $g_1$ and let $n$ be the arrow making the following diagram commute.

$$\begin{tikzcd}
A \arrow[dr, "n"] \arrow[drr, bend left, "a_2"] \arrow[ddr, bend right, "a_1"'] & & \\
& M \arrow[r, "u_2"] \arrow[d, "u_1"] \arrow[dr, phantom, "\lrcorner", very near start] & G \arrow[d, "g_1"] \\
& F \arrow[r, "f_2"'] & A
\end{tikzcd}$$

Let $ip$ be the image factorisation of $M \xrightarrow{(f_1u_1, g_2u_2)} A \times A$. Since the image factorisation is functorial (proposition 2.22) and $(f_1u_1, g_2u_2)n = (1_A, 1_A)$, there is a morphism $n'$ making the diagram

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "n"'] & A \arrow[r, "{(1_A,1_A)}"] \arrow[d, "n'"] & A \times A \arrow[d, equal] \\
M \arrow[r, "p"'] & I \arrow[r, "i"'] & A \times A
\end{tikzcd}$$

commute. So, if we set $i = (i_1, i_2)$, the relation $I \overset{i_1}{\underset{i_2}{\rightrightarrows}} A$ is reflexive. But since $\mathcal{A}$ is Mal'cev (proposition 2.26) and Barr exact, there exists a morphism $A \xrightarrow{t} D$ such that

$I \underset{i_2}{\overset{i_1}{\rightrightarrows}} A$ is the kernel pair of $t$. Moreover, by lemma 2.19, we can suppose $t$ to be the coequalizer of $i_1$ and $i_2$. Consider now the map $G \xrightarrow{k} F$ making the diagram

$$\begin{tikzcd}
G \arrow[dr, "k"] \arrow[drr, bend left, "g_1"] \arrow[ddr, bend right, "g_1"'] & & \\
& F \arrow[r, "f_2"] \arrow[d, "f_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\
& A \arrow[r, "f"'] & B
\end{tikzcd}$$

commute. Since $f_2k = g_1$, there is a morphism $G \xrightarrow{l} M$ such that the following diagram commutes.

$$\begin{tikzcd}
G \arrow[dr, "l"] \arrow[drr, bend left, "1_G"] \arrow[ddr, bend right, "k"'] & & \\
& M \arrow[r, "u_2"] \arrow[d, "u_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & G \arrow[d, "g_1"] \\
& F \arrow[r, "f_2"'] & A
\end{tikzcd}$$

Hence the morphism $G \xrightarrow{l} M \xrightarrow{(f_1u_1,g_2u_2)} A \times A$ is nothing but $(g_1, g_2)$. Thus $g_1 = i_1pl$ and $g_2 = i_2pl$. So, $tg_1 = tg_2$. Recall from definition 2.21 that, since $g$ is a regular epimorphism, it is the coequalizer of its kernel pair. Consequently, we can find a map $C \xrightarrow{r} D$ such that $rg = t$. Similarly, there is a map $B \xrightarrow{s} D$ such that $sf = t$.

$$\begin{tikzcd}
G \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] & A \arrow[r, "g"] \arrow[dr, "t"'] & C \arrow[d, "r"] & F \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_2"'] & A \arrow[r, "f"] \arrow[dr, "t"'] & B \arrow[d, "s"] \\
& & D & & & D
\end{tikzcd}$$

Let's prove that

$$\begin{tikzcd}
A \arrow[r, "g"] \arrow[d, "f"'] & C \arrow[d, "r"] \\
B \arrow[r, "s"'] & D
\end{tikzcd} \tag{2.2}$$

is a pushout square. Suppose that $B \xrightarrow{x} Z$ and $C \xrightarrow{y} Z$ are two maps such that $xf = yg$. So we can compute

$$\begin{aligned}
ygi_1p &= xff_1u_1 \\
&= xff_2u_1 \\
&= ygg_1u_2 \\
&= ygg_2u_2 \\
&= ygi_2p.
\end{aligned}$$

This implies $ygi_1 = ygi_2$. Keeping in mind that $t$ is the coequalizer of $i_1$ and $i_2$, we know there is a unique map $D \xrightarrow{z} Z$ such that $zt = yg$. But this occurs if and only if

$zrg = yg$ and $zsf = xf$ which happens if and only if $zr = y$ and $zs = x$. Therefore, the square (2.2) is a pushout. So we can assume $C \xrightarrow{q_1} Q = C \xrightarrow{r} D$ and $B \xrightarrow{q_2} Q = B \xrightarrow{s} D$. Now we shall use the fact that a composition of two regular epimorphisms is a regular epimorphism (corollary 2.30) and the well-known property saying that if a rectangle diagram is made of small pullback squares, then it is a pullback (see proposition 2.5.9 in [2]). So, consider the following diagram where all rectangles are pullbacks.

$$\begin{tikzcd}
I \arrow[r, "v_2'"] \arrow[d, "v_1'"'] \arrow[dr, phantom, "\lrcorner", very near start] & W \arrow[r, "w_2"] \arrow[d, "w_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "g"] \\
V \arrow[r, "v_2"] \arrow[d, "v_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & P \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "r"] \\
A \arrow[r, "f"'] & B \arrow[r, "s"'] & D
\end{tikzcd}$$

Since $sf = rg = t$, $i_1 = v_1 v_1'$ and $i_2 = w_2 v_2'$. Finally, if we keep in mind that pullbacks preserve regular epimorphisms, we can compute

$$\begin{aligned}
\operatorname{im}(f,g) &= \operatorname{im}(fg_1u_2, gg_1u_2) \\
&= \operatorname{im}(ff_2u_1, gg_1u_2) \\
&= \operatorname{im}(ff_1u_1, gg_2u_2) \\
&= \operatorname{im}(fi_1p, gi_2p) \\
&= \operatorname{im}(fi_1, gi_2) \\
&= \operatorname{im}(fv_1v_1', gw_2v_2') \\
&= \operatorname{im}(p_1w_1v_2', p_2w_1v_2') \\
&= \operatorname{im}(p_1, p_2) \\
&= (p_1, p_2),
\end{aligned}$$

which proves that $(f,g) = (p_1,p_2)e$ is the image factorisation of $A \xrightarrow{(f,g)} B \times C$. Therefore $e$ is a regular epimorphism.

$\square$

Now we are able to prove that, if $f = pi$ (rather than $ip$ in the image factorisation) with $i$ and $p$ normal, then $f$ is proper.

**Lemma 2.43.** Let $f \in \mathcal{A}(A,B)$ where $\mathcal{A}$ is a semi-abelian category. If $f = p'i'$ where $p'$ is a normal epimorphism and $i'$ a normal monomorphism, then $f$ is a proper morphism.

*Proof.* Let $f = ip$ be the image factorisation of $f$. We have to show that $i$ is a normal monomorphism.

$$\begin{tikzcd}
A \arrow[r, "p"] \arrow[d, "i'"'] & I \arrow[d, "i"] \\
I' \arrow[r, "p'"'] & B
\end{tikzcd}$$

Let $e = \operatorname{coker} i' \in \mathcal{A}(I', C)$. Since $i'$ is a normal monomorphism, $i' = \ker e$. Consider $Q$

the pushout of $e$ and $p'$. Denote by $P$ the pullback of this pushout.

$$\begin{tikzcd}
I' \arrow[r, "p'"] \arrow[d, "e"'] & B \arrow[d, "q_1"] \\
C \arrow[r, "q_2"'] \arrow[ur, phantom, "\ulcorner" very near end] & Q
\end{tikzcd}
\qquad
\begin{tikzcd}
P \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\lrcorner" very near start] & B \arrow[d, "q_1"] \\
C \arrow[r, "q_2"'] & Q
\end{tikzcd}$$

Let $I' \xrightarrow{h} P$ be the unique map such that $p_1h = p'$ and $p_2h = e$. By lemma 2.42, $h$ is a regular epimorphism. Now let $k = \ker p_2 \in \mathcal{A}(K, P)$. We are going to prove that $p_1k = \ker q_1$. First notice that $q_1p_1k = q_2p_2k = 0$. Now suppose that there is a map $Z \xrightarrow{z} B$ such that $q_1z = 0$. So we know the existence of a morphism $Z \xrightarrow{m} P$ such that $p_1m = z$ and $p_2m = 0$.

$$\begin{tikzcd}
Z \arrow[dr, "m"] \arrow[drr, bend left, "z"] \arrow[ddr, bend right, "0"'] & & \\
& P \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\lrcorner" very near start] & B \arrow[d, "q_1"] \\
& C \arrow[r, "q_2"'] & Q
\end{tikzcd}$$

Therefore there is a map $Z \xrightarrow{n} K$ satisfying $kn = m$ and so $p_1kn = z$. But if $p_1kx = p_1ky$ for some maps $x$ and $y$, this implies $kx = ky$ since $p_2kx = 0 = p_2ky$ and $P$ is a pullback. So $x = y$ and $p_1k$ is a monomorphism. So there is only one $n$ with $p_1kn = z$ and we can conclude that $p_1k = \ker q_1$. Now, notice that $q_1ip = q_1p'i' = q_2ei' = 0$. Thus $q_1i = 0$. So there is a map $t$ making

$$\begin{tikzcd}
I \arrow[dr, "t"] \arrow[drr, bend left, "i"] \arrow[ddr, bend right] & & \\
& K \arrow[r, "p_1k"] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & B \arrow[d, "q_1"] \\
& 0 \arrow[r] & Q
\end{tikzcd}$$

commute and $t$ is a monomorphism as $i$ is. But $p_1ktp = ip = p'i' = p_1hi'$ and $p_2ktp = 0 = ei' = p_2hi'$. Consequently, $ktp = hi'$.

$$\begin{tikzcd}
A \arrow[r, "i'"] \arrow[d, "tp"'] & I' \arrow[r, "e"] \arrow[d, "h"] & C \arrow[d, equal] \\
K \arrow[r, "k"'] & P \arrow[r, "p_2"'] & C
\end{tikzcd}$$

Since $1_C$ is a monomorphism, by lemma 4.2.4 in [4], we know that the left-hand square is a pullback. But $h$ is a regular epimorphism and pullbacks preserve them in a semi-abelian category. So $tp$ is a regular epimorphism. Therefore $t$ is a regular epimorphism. But since it is a monomorphism, it is a isomorphism. So $i = \ker q_1$ which is a normal monomorphism. □

Now, let's define the homology of a proper chain complex.

**Definition 2.44.** Let $C_\bullet = \cdots \longrightarrow C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \longrightarrow \cdots$ be a proper chain complex in a semi-abelian category $\mathcal{A}$. If $d'_{n+1}$ is the unique morphism such that $\ker d_n \circ d'_{n+1} = d_{n+1}$, we denote by $H_n(C_\bullet)$ the object $\operatorname{Coker} d'_{n+1}$ and we call it the $\mathbf{n^{th}}$ **homology of** $C_\bullet$.

$$\begin{tikzcd}
C_{n+1} \arrow[rr, "d_{n+1}"] \arrow[dr, "d'_{n+1}"'] & & C_n \arrow[r, "d_n"] & C_{n-1} \\
& \operatorname{Ker} d_n \arrow[ur, "\ker d_n"'] \arrow[d, "\operatorname{coker} d'_{n+1}"] & & \\
& H_n(C_\bullet) & &
\end{tikzcd} \tag{2.3}$$

If $C_\bullet \xrightarrow{f} D_\bullet$ ,

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "f_{n+1}"'] & C_n \arrow[r, "d_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r] & D_{n+1} \arrow[r, "e_{n+1}"'] & D_n \arrow[r, "e_n"'] & D_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

we can define $H_n(f) \in \mathcal{A}(H_n(C_\bullet), H_n(D_\bullet))$ as the unique map such that $H_n(f) \circ \operatorname{coker} d'_{n+1} = \operatorname{coker} e'_{n+1} \circ Z_n f$ where $Z_n f$ is the map such that $\ker e_n \circ Z_n f = f_n \circ \ker d_n$. So we have defined a functor $H_n : \mathrm{PCh}\,\mathcal{A} \to \mathcal{A}$ for all $n \in \mathbb{Z}$.

We could have defined the homology dually.

**Lemma 2.45.** Let $C_\bullet$ be a proper chain complex in a semi-abelian category. If $d''_n$ is the unique map such that $d''_n \circ \operatorname{coker} d_{n+1} = d_n$, then $\operatorname{Ker} d''_n = H_n(C_\bullet)$.

$$\begin{tikzcd}
C_{n+1} \arrow[rr, "d_{n+1}"] \arrow[dr, "d'_{n+1}"'] & & C_n \arrow[rr, "d_n"] \arrow[dr, "\operatorname{coker} d_{n+1}"] & & C_{n-1} \\
& \operatorname{Ker} d_n \arrow[ur, "\ker d_n"] \arrow[d, "\operatorname{coker} d'_{n+1}"] & & \operatorname{Coker} d_{n+1} \arrow[ur, "d''_n"'] & \\
& H_n(C_\bullet) \arrow[rr, equal] & & H_n(C_\bullet) \arrow[u, "\ker d''_n"] &
\end{tikzcd} \tag{2.4}$$

*Proof.* First, notice that $d''_n \circ \operatorname{coker} d_{n+1} \circ \ker d_n = 0$, so there is a unique morphism $\operatorname{Ker} d_n \xrightarrow{s} \operatorname{Ker} d''_n$ such that $\ker d''_n \circ s = \operatorname{coker} d_{n+1} \circ \ker d_n$. Similarly, there is a unique arrow $H_n(C_\bullet) \xrightarrow{t} \operatorname{Coker} d_{n+1}$ such that $t \circ \operatorname{coker} d'_{n+1} = \operatorname{coker} d_{n+1} \circ \ker d_n$. Moreover, since $d''_n \circ t \circ \operatorname{coker} d'_{n+1} = d''_n \circ \operatorname{coker} d_{n+1} \circ \ker d_n = 0$, we know that $d''_n \circ t = 0$ and so there is a unique map $H_n(C_\bullet) \xrightarrow{u} \operatorname{Ker} d''_n$ such that $\ker d''_n \circ u = t$. In addition, $\ker d''_n \circ u \circ \operatorname{coker} d'_{n+1} = t \circ \operatorname{coker} d'_{n+1} = \operatorname{coker} d_{n+1} \circ \ker d_n = \ker d''_n \circ s$ and therefore

$u \circ \operatorname{coker} d'_{n+1} = s$. We have to prove that $u$ is an isomorphism.

$$\begin{tikzcd}
C_{n+1} \arrow[rr, "d_{n+1}"] \arrow[dr, "d'_{n+1}"'] & & C_n \arrow[rr, "d_n"] \arrow[dr, "\operatorname{coker} d_{n+1}"] & & C_{n-1} \\
& \operatorname{Ker} d_n \arrow[ur, "\ker d_n"] \arrow[d, "\operatorname{coker} d'_{n+1}"'] \arrow[drr, dotted, "s"] & & \operatorname{Coker} d_{n+1} \arrow[ur, "d''_n"'] & \\
& H_n(C_\bullet) \arrow[urr, dotted, "t"] \arrow[rr, dotted, "u"'] & & \operatorname{Ker} d''_n \arrow[u, "\ker d''_n"'] &
\end{tikzcd}$$

If there are two maps $Z \xrightarrow{x} C_n$ and $Z \xrightarrow{y} \operatorname{Ker} d''_n$ with $\operatorname{coker} d_{n+1} \circ x = \ker d''n \circ y$, then $d_n x = d''_n \circ \operatorname{coker} d_{n+1} \circ x = d''n \circ \ker d''n \circ y = 0$. So there is a unique map $Z \xrightarrow{m} \operatorname{Ker} d_n$ such that $\ker d_n \circ m = x$. Moreover, $sm = y$ since $\ker d''_n \circ s \circ m = \operatorname{coker} d_{n+1} \circ \ker d_n \circ m = \operatorname{coker} d_{n+1} \circ x = \ker d''_n \circ y$. So, on the following diagram, the left-hand square is a pullback.

$$\begin{tikzcd}
\operatorname{Ker} d_n \arrow[r, "\ker d_n"] \arrow[d, "s"'] & C_n \arrow[r, "d_n"] \arrow[d, "\operatorname{coker} d_{n+1}"] & C_{n-1} \arrow[d, equal] \\
\operatorname{Ker} d''_n \arrow[r, "\ker d''_n"] & \operatorname{Coker} d_{n+1} \arrow[r, "d''_n"] & C_{n-1}
\end{tikzcd}$$

Thus $s$ is a regular epimorphism since pullbacks preserve them in a semi-abelian category. So $u$ is also a regular epimorphism. So, it remains to prove that $u$ is a monomorphism. To do so, it is enough to prove that $t$ is a monomorphism. Let $d_{n+1} = ip$ and $d'_{n+1} = i'p'$ be the image factorisations of $d_{n+1}$ and $d'_{n+1}$. By functoriality of the image factorisation (proposition 2.22), there is a map $j$ making the following diagram commute.

$$\begin{tikzcd}
C_{n+1} \arrow[r, "p'"] \arrow[d, equal] & I' \arrow[r, "i'"] \arrow[d, "j"] & \operatorname{Ker} d_n \arrow[r, "\operatorname{coker} d'_{n+1}"] \arrow[d, "\ker d_n"] & H_n(C_\bullet) \arrow[d, "t"] \\
C_{n+1} \arrow[r, "p"'] & I \arrow[r, "i"'] & C_n \arrow[r, "\operatorname{coker} d_{n+1}"'] & \operatorname{Coker} d_{n+1}
\end{tikzcd}$$

Because $jp' = p$, $j$ is a regular epimorphism. But since $ij = \ker d_n \circ i'$, $j$ is a monomorphism and thus a isomorphism. We also know that $d_{n+1}$ is a proper morphism. So $i$ is a normal monomorphism. So by lemma 2.41, $i'$ is also a normal monomorphism. But since $p$ and $p'$ are epimorphisms, $\operatorname{coker} d_{n+1} = \operatorname{coker} i$ and $\operatorname{coker} d'_{n+1} = \operatorname{coker} i'$. So, we have the following commutative diagram where the two rows are exact.

$$\begin{tikzcd}
0 \arrow[r] & I' \arrow[r, "i'"] \arrow[d, "j"] & \operatorname{Ker} d_n \arrow[r, "\operatorname{coker} i'"] \arrow[d, "\ker d_n"] & H_n(C_\bullet) \arrow[r] \arrow[d, "t"] & 0 \\
0 \arrow[r] & I \arrow[r, "i"'] & C_n \arrow[r, "\operatorname{coker} i"'] & \operatorname{Coker} i \arrow[r] & 0
\end{tikzcd}$$

Now, consider the pullback $Q$ of $t$ and $\operatorname{coker} i$. By its universal property, there is a morphism $a$ making the following diagram commute.

$$\begin{tikzcd}
0 \arrow[r] & I' \arrow[rr, "i'"] \arrow[dd, "j"] & & \operatorname{Ker} d_n \arrow[rr, "\operatorname{coker} i'"] \arrow[dd, "\ker d_n"'] \arrow[dr, "a"] & & H_n(C_\bullet) \arrow[rr] \arrow[dd, "t"] \arrow[dr, equal] & & 0 \\
& & \operatorname{Ker} q_1 \arrow[rr, "\ker q_1"] & & Q \arrow[rr, "q_1"] \arrow[dl, "q_2"] \arrow[dr, phantom, very near start, "\lrcorner"] & & H_n(C_\bullet) \arrow[dl, "t"] & \\
0 \arrow[r] & I \arrow[rr, "i"'] & & C_n \arrow[rr, "\operatorname{coker} i"'] & & \operatorname{Coker} i \arrow[rr] & & 0
\end{tikzcd}$$

Since $\operatorname{coker} i \circ q_2 \circ \ker q_1 = t \circ q_1 \circ \ker q_1 = 0$, there is a morphism $\operatorname{Ker} q_1 \xrightarrow{c} I$ such that $ic = q_2 \ker q_1$. By a classic result about kernels, (see lemma 4.2.4 in [4]), we know that $c$ is an isomorphism. In addition, since $q_1 a i' = 0$, there exists a map $I' \xrightarrow{b} \operatorname{Ker} q_1$ such that $\ker q_1 \circ b = ai'$. Moreover, $cb = j$ since $icb = q_2 \circ \ker q_1 \circ b = q_2 a i' = \ker d_n \circ i' = ij$.

$$\begin{tikzcd}
0 \arrow[r] & I' \arrow[rr, "i'"] \arrow[dd, "j"] \arrow[dr, "b"] & & \operatorname{Ker} d_n \arrow[rr, "\operatorname{coker} i'"] \arrow[dd, "\ker d_n"'] \arrow[dr, "a"] & & H_n(C_\bullet) \arrow[rr] \arrow[dd, "t"] \arrow[dr, equal] & & 0 \\
& & \operatorname{Ker} q_1 \arrow[rr, "\ker q_1"] \arrow[dl, "c"] & & Q \arrow[rr, "q_1"] \arrow[dl, "q_2"] \arrow[dr, phantom, very near start, "\lrcorner"] & & H_n(C_\bullet) \arrow[dl, "t"] & \\
0 \arrow[r] & I \arrow[rr, "i"'] & & C_n \arrow[rr, "\operatorname{coker} i"'] & & \operatorname{Coker} i \arrow[rr] & & 0
\end{tikzcd}$$

So $b$ is an isomorphism and $q_1$ is a regular epimorphism since $\operatorname{coker} i$ is. By the regular Short Five Lemma, $a$ is also an isomorphism. Therefore, $q_2$ is a monomorphism and the same holds for $t$ since pullbacks reflect monomorphisms (see lemma 2.33).

□

Due to this lemma, we could have also defined the action of $H_n$ on the arrows of $\mathrm{PCh}\,\mathcal{A}$ dually.

The following proposition explains why homology 'measures' exactness.

**Proposition 2.46.** Let $C_\bullet$ be a proper chain complex in a semi-abelian category. $C_\bullet$ is exact at $C_n$ if and only if $H_n(C_\bullet) = 0$.

*Proof.* Let $d_{n+1} = ip$ be the image factorisation of $d_{n+1}$. So $d_n i = 0$ and there is a unique map $m$ such that $\ker d_n \circ m = i$. Since $i$ is a monomorphism, $m$ is also a monomorphism. Moreover, we know that $d_{n+1}$ is a proper morphism. So $i$ is a normal monomorphism and by lemma 2.41, $m$ is a normal monomorphism. In addition, we know that $mp = d'_{n+1}$. But since $p$ is an epimorphism, $\operatorname{coker} d'_{n+1} = \operatorname{coker}(mp) = \operatorname{coker} m$.

$$\begin{tikzcd}
C_{n+1} \arrow[rrr, "d_{n+1}"] \arrow[dr, "p"] & & & C_n \arrow[r, "d_n"] & C_{n-1} \\
& I \arrow[r, "m"] \arrow[urr, bend left, "i"] & \operatorname{Ker} d_n \arrow[ur, "\ker d_n"'] \arrow[dr, "\operatorname{coker} m"] & & \\
& & & H_n(C_\bullet) &
\end{tikzcd}$$

$C_\bullet$ is exact at $C_n$ if and only if $i = \ker d_n$, i.e. if and only if $m$ is an isomorphism. But if $m$ is an isomorphism, $H_n(C_\bullet) = \operatorname{Coker} m = 0$. Conversely, if $\operatorname{Coker} m = 0$, since $m$ is a normal monomorphism, $m = \ker(\operatorname{coker} m) = \ker(\ \operatorname{Ker} d_n \longrightarrow 0\ )$ which is an isomorphism.

□

# 3 Simplicial Objects

At the end of chapter 2, we defined the homology of a proper chain complex. In this chapter, we are going to introduce simplicial objects. As we shall see, these objects induce a proper chain complex, and therefore homology functors. The aim here is to prove a Comparison Theorem to create maps between simplicial objects and compare their image under the induced homology functors.

## 3.1 Simplicial objects and their homology

In this section, we define simplicial objects and their induced homology. We also give the first properties of this homology. For any natural number $n \in \mathbb{N}$, let us fix the set $[n]$ to be $[n] = \{0, \ldots, n\}$.

**Definition 3.1.** In a category $\mathcal{A}$, a **simplicial object** $\mathbb{A}$ is the data of a sequence $(A_n)_{n \in \mathbb{N}} \subset \operatorname{ob} \mathcal{A}$ together with morphisms $A_n \xrightarrow{\partial_i} A_{n-1}$ for each $i \in [n]$ and $n \in \mathbb{N}_0$ called **face operators**

$$\begin{tikzcd}\cdots \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & A_2 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & A_1 \arrow[r, shift left] \arrow[r, shift right] & A_0\end{tikzcd}$$

and with morphisms $A_n \xrightarrow{\sigma_i} A_{n+1}$ for each $i \in [n]$ and $n \in \mathbb{N}$ named **degeneracy operators** satisfying

$$\partial_i \circ \partial_j = \partial_{j-1} \circ \partial_i \quad \text{if } i < j, \tag{3.1}$$

$$\sigma_i \circ \sigma_j = \sigma_{j+1} \circ \sigma_i \quad \text{if } i \leqslant j, \tag{3.2}$$

$$\partial_i \circ \sigma_j = \begin{cases} \sigma_{j-1} \circ \partial_i & \text{if } i < j \\ 1_{A_n} & \text{if } i = j \text{ or } i = j+1 \\ \sigma_j \circ \partial_{i-1} & \text{if } i > j+1 \end{cases} \tag{3.3}$$

called the **simplicial identities**. We write $\mathrm{S}\,\mathcal{A}$ for the category of simplicial objects where a morphism $\mathbb{A} \xrightarrow{f} \mathbb{B}$ is the data of $f_n \in \mathcal{A}(A_n, B_n)$ for all $n \in \mathbb{N}$ such that $f_{n-1} \circ \partial_i = \partial'_i \circ f_n$ for all $n \in \mathbb{N}_0$ and $i \in [n]$ and $f_n \circ \sigma_i = \sigma'_i \circ f_{n-1}$ for all $n \in \mathbb{N}_0$ and $i \in [n-1]$. Morphisms in $\mathrm{S}\,\mathcal{A}$ are also called **simplicial maps**. If the $f_n$'s commute appropriately only with the face operators (the $\partial_i$'s), we say it is a **semi-simplicial map**, whereas the induced subcategory is denoted by $\mathrm{S}'\,\mathcal{A}$.

We can also define some particular elements in $\mathrm{S}\,\mathcal{A}$.

**Definition 3.2.** An **augmented simplicial object** is a simplicial object $\mathbb{A}$ with an additional map $A_0 \xrightarrow{\partial_0} A_{-1}$ such that $\partial_0 \circ \partial_0 = \partial_0 \circ \partial_1$. An **augmented simplicial map** between two augmented simplicial objects is the data of $f_n \in \mathcal{A}(A_n, B_n)$ for all $n \geqslant -1$ which is a simplicial map and such that $f_{-1} \circ \partial_0 = \partial'_0 \circ f_0$. This form a subcategory of $\mathrm{S}\,\mathcal{A}$ denoted by $\mathrm{AS}\,\mathcal{A}$. As above, if the $f_n$'s commute only with the face operators, we call it an **augmented semi-simplicial map** and we write $\mathrm{AS}'\,\mathcal{A}$ for the corresponding subcategory.

A **right-contractible augmented simplicial object** is an augmented simplicial object for which there exist maps $A_n \xrightarrow{h_n} A_{n+1}$ for $n \geqslant -1$ such that $\partial_{n+1} \circ h_n = 1_{A_n}$ and $\partial_i \circ h_n = h_{n-1} \circ \partial_i$ for all $i \in [n]$.

$$\begin{tikzcd}
\cdots \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & A_2 \arrow[l, bend left=60, "h_2"] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & A_1 \arrow[l, bend left=60, "h_1"] \arrow[r, shift left] \arrow[r, shift right] & A_0 \arrow[l, bend left=60, "h_0"] \arrow[r] & A_{-1} \arrow[l, bend left=60, "h_{-1}"]
\end{tikzcd}$$

Similarly, a **left-contractible augmented simplicial object** is an augmented simplicial object for which we can find morphisms $A_n \xrightarrow{h_n} A_{n+1}$ for $n \geqslant -1$ such that $\partial_0 \circ h_n = 1_{A_n}$ and $\partial_i \circ h_n = h_{n-1} \circ \partial_{i-1}$ for all $i \in \{1, \ldots, n+1\}$.

Given a simplicial object $\mathbb{A}$ in a semi-abelian category $\mathcal{A}$, let us write $N_{-n}\mathbb{A} = 0$ for $n \in \mathbb{N}_0$, $N_0\mathbb{A} = A_0$ and $N_n\mathbb{A} = \operatorname{Ker}\ A_n \xrightarrow{(\partial_j)_{j\in[n-1]}} A^n_{n-1}$ for $n \in \mathbb{N}_0$. Thus, if $n \geqslant 2$, the map $N_n\mathbb{A} \xrightarrow{\ker(\partial_j)_{j\in[n-1]}} A_n \xrightarrow{\partial_n} A_{n-1} \xrightarrow{(\partial_j)_{j\in[n-2]}} A^{n-1}_{n-2}$ is 0 since for all $i \in [n-2]$,

$$\begin{aligned}
\pi_i \circ (\partial_j)_{j\in[n-2]} \circ \partial_n \circ \ker(\partial_j)_{j\in[n-1]} &= \partial_i \circ \partial_n \circ \ker(\partial_j)_{j\in[n-1]} \\
&= \partial_{n-1} \circ \partial_i \circ \ker(\partial_j)_{j\in[n-1]} \\
&= \partial_{n-1} \circ \pi'_i \circ (\partial_j)_{j\in[n-1]} \circ \ker(\partial_j)_{j\in[n-1]} \\
&= 0.
\end{aligned}$$

Therefore, there is a unique map $N_n\mathbb{A} \xrightarrow{d_n} N_{n-1}\mathbb{A}$ such that $\ker(\partial_j)_{j\in[n-2]} \circ d_n = \partial_n \circ \ker(\partial_j)_{j\in[n-1]}$. Hence, we can define the Moore complex as follow.

**Definition 3.3.** If $\mathbb{A}$ is a simplicial object in a semi-abelian category, we write $N_{-n}\mathbb{A} = 0$, $N_0\mathbb{A} = A_0$ and $N_n\mathbb{A} = \operatorname{Ker}\ A_n \xrightarrow{(\partial_j)_{j\in[n-1]}} A^n_{n-1}$ for $n \in \mathbb{N}_0$. The **Moore complex** (or **normalised chain complex**) is the chain $\cdots \xrightarrow{d_{n+1}} N_n\mathbb{A} \xrightarrow{d_n} N_{n-1}\mathbb{A} \xrightarrow{d_{n-1}} \cdots$ where $d_n$ is the map induced by $\partial_n \circ \ker(\partial_j)_{j\in[n-1]}$ if $n > 0$. We denote the Moore complex of $\mathbb{A}$ by $N(\mathbb{A})$.

As expected, the Moore complex is a proper chain complex.

**Lemma 3.4.** If $\mathbb{A}$ is a simplicial object in a semi-abelian category $\mathcal{A}$, then $N(\mathbb{A})$ is a proper chain complex.

*Proof.* In one hand, we have to prove that $d_n \circ d_{n+1} = 0$ for all $n \in \mathbb{Z}$. Since it is trivial for $n \leqslant 0$, we can suppose $n > 0$. For $n \geqslant 2$, we can prove it by showing that $\ker(\partial_j)_{j\in[n-2]} \circ d_n \circ d_{n+1} = 0$, which can be done by direct computations:

$$\begin{aligned}
\ker(\partial_j)_{j\in[n-2]} \circ d_n \circ d_{n+1} &= \partial_n \circ \ker(\partial_j)_{j\in[n-1]} \circ d_{n+1} \\
&= \partial_n \circ \partial_{n+1} \circ \ker(\partial_j)_{j\in[n]} \\
&= \partial_n \circ \partial_n \circ \ker(\partial_j)_{j\in[n]} \\
&= 0.
\end{aligned}$$

For $n = 1$, we can use the same reasoning since $d_1 \circ d_2 = \partial_1 \circ \ker \partial_0 \circ d_2$.

In another hand, we have to prove that $d_n$ is a proper morphism for all $n \in \mathbb{Z}$. If $n \leqslant 0$, $N_n\mathbb{A} \xrightarrow{d_n} 0$ is a normal epimorphism, so it is proper. Now, if $n > 0$, we know that $\partial_n \circ \sigma_{n-1} = 1_{A_n}$, so $\partial_n$ is a split epimorphism and thus a regular epimorphism. Since $d_1 = \partial_1 \circ \ker \partial_0$ and $\ker (\partial_j)_{j\in[n-2]} \circ d_n = \partial_n \circ \ker (\partial_j)_{j\in[n-1]}$ for $n > 1$, by lemma 2.41, it is enough to prove that $\partial_n \circ \ker (\partial_j)_{j\in[n-1]}$ is proper. But this is straight forward by lemma 2.43.

□

Notice that if we have a semi-simplicial map $\mathbb{A} \xrightarrow{f} \mathbb{B}$, it induces a morphism $N(f) \in \mathrm{PCh}(N(\mathbb{A}), N(\mathbb{B}))$. Indeed, let $(N(f))_0 = f_0$ and for $n > 0$, let $(N(f))_n$ be the map making

$$\begin{tikzcd}
N_n(\mathbb{A}) \arrow[r] \arrow[d, "(N(f))_n"'] & A_n \arrow[r, "(\partial_j)_{j\in[n-1]}"] \arrow[d, "f_n"] & A_{n-1}^n \arrow[d, "\overline{f_{n-1}}"] \\
N_n(\mathbb{B}) \arrow[r] & B_n \arrow[r, "(\partial'_j)_{j\in[n-1]}"'] & B_{n-1}^n
\end{tikzcd}$$

commute, where $\overline{f_{n-1}}$ is the unique map satisfying $\pi'_i \circ \overline{f_{n-1}} = f_{n-1} \circ \pi_i$ for all $i \in [n-1]$. It is a well-defined map in $\mathrm{PCh}\,\mathcal{A}$ because $(N(f))_n \circ d_{n+1} = d'_{n+1} \circ (N(f))_{n+1}$ since these two maps are equal if we compose them with $\ker (\partial'_j)_{j\in[n-1]}$. In particular, this construction turns $N$ into a functor $\mathrm{S}\,\mathcal{A} \to \mathrm{PCh}\,\mathcal{A}$.

Moreover, due to the previous lemma, we can define the homology sequence of $N(\mathbb{A})$.

**Definition 3.5.** If $\mathbb{A}$ is a simplicial object in a semi-abelian category, we write $H_n\mathbb{A}$ for $H_nN(\mathbb{A})$. Moreover, if $\mathbb{A} \xrightarrow{f} \mathbb{B}$ is a semi-simplicial map, we shall write $H_nf$ for $H_nN(f)$ if there is not any risk of confusion.

The following proposition gives us a better understanding of $H_0\mathbb{A}$.

**Proposition 3.6.** Let $\mathbb{A}$ be a simplicial object in a semi-abelian category $\mathcal{A}$. Then, $H_0\mathbb{A}$ is the coequalizer of $\begin{tikzcd} A_1 \arrow[r, shift left, "\partial_0"] \arrow[r, shift right, "\partial_1"'] & A_0 \end{tikzcd}$.

*Proof.* We have to compute the homology at $A_0$ of the sequence,

$$\begin{tikzcd}
\mathrm{Ker}\,\partial_0 \arrow[r, "\partial_1 \ker \partial_0"] & A_0 \arrow[r] & 0
\end{tikzcd}$$

which is, by the dual definition of homology (lemma 2.45), nothing but $\mathrm{Coker}(\partial_1 \ker \partial_0)$. Let $c$ be the coequalizer of $\partial_0$ and $\partial_1$. So, we have to prove that $c = \mathrm{coker}(\partial_1 \ker \partial_0)$.

$$\begin{tikzcd}
A_1 \arrow[r, shift left, "\partial_0"] \arrow[r, shift right, "\partial_1"'] & A_0 \arrow[r, "c"] & C
\end{tikzcd}$$

First, notice that $c\partial_1 \ker \partial_0 = c\partial_0 \ker \partial_0 = 0$. Now, consider a map $A_0 \xrightarrow{f} D$ such that $f\partial_1 \ker \partial_0 = 0$. By definition of $c$, we only have to show that $f\partial_0 = f\partial_1$. Let's recall from the definition of simplicial object that $\partial_0\sigma_0 = \partial_1\sigma_0 = 1_{A_0}$. Hence $\partial_0$ is a split epimorphism, so it is a normal epimorphism since regular ones are normal (proposition

2.31). Therefore $\partial_0 = \operatorname{coker}(\ker \partial_0)$.

$$\begin{tikzcd}
\operatorname{Ker}\partial_0 \arrow[r, "\ker\partial_0"] \arrow[d] & A_1 \arrow[d, "\partial_0"] \arrow[rdd, bend left, "f\partial_1"] & \\
0 \arrow[r] \arrow[rrd, bend right] & A_0 \arrow[rd, "m"] \arrow[ul, phantom, "\ulcorner", very near start] & \\
 & & D
\end{tikzcd}$$

Since $f\partial_1 \ker\partial_0 = 0$, there is a map $m$ such that $m\partial_0 = f\partial_1$. Consequently, $m = m\partial_0\sigma_0 = f\partial_1\sigma_0 = f$ and $f\partial_0 = f\partial_1$, which concludes the proof.

□

We can even have a complete description of $H_n\mathbb{A}$ if we consider a right-contractible augmented simplicial object. In order to do so, let us prove a lemma, quite trivial, but which will also be useful in section 3.2.

**Lemma 3.7.** Let $\mathbb{A}$ be a simplicial object in a semi-abelian category $\mathcal{A}$. Then, for all $n \in \mathbb{N}_0$,

$$\ker\left(\partial_j\right)_{j\in[n-1]} \circ \ker d_n = \ker\left(\partial_j\right)_{j\in[n]}.$$

*Proof.* First, recall that, if $n \geqslant 2$, then $\ker\left(\partial_j\right)_{j\in[n-2]} \circ d_n = \partial_n \circ \ker\left(\partial_j\right)_{j\in[n-1]}$, hence, for all $n \in \mathbb{N}_0$,

$$\ker d_n = \ker\left(\partial_n \circ \ker\left(\partial_j\right)_{j\in[n-1]}\right). \tag{3.4}$$

So, $\left(\partial_j\right)_{j\in[n]} \circ \ker\left(\partial_j\right)_{j\in[n-1]} \circ \ker d_n = 0$. Now, suppose we have a map $f \in \mathcal{A}(B, A_n)$ such that $\left(\partial_j\right)_{j\in[n]} \circ f = 0$.

$$\begin{tikzcd}
B \arrow[rrd, bend left, "f"] \arrow[rdd, bend right] & & \\
 & \operatorname{Ker} d_n \arrow[r, "\ker\left(\partial_j\right)_{j\in[n-1]}\circ\ker d_n"] \arrow[d] & A_n \arrow[d, "\left(\partial_j\right)_{j\in[n]}"] \\
 & 0 \arrow[r] & A^{n+1}_{n-1}
\end{tikzcd}$$

We know that $f$ can be written as $\ker\left(\partial_j\right)_{j\in[n-1]} \circ f'$ for a $f' \in \mathcal{A}(B, N_n\mathbb{A})$ because $\left(\partial_j\right)_{j\in[n-1]} \circ f = 0$. But since $\partial_n \circ \ker\left(\partial_j\right)_{j\in[n-1]} \circ f' = 0$ and (3.4), $f'$ factors through $\ker d_n$. Therefore $f$ factors uniquely through $\ker\left(\partial_j\right)_{j\in[n-1]} \circ \ker d_n$, which concludes the proof.

□

**Proposition 3.8.** Let $\mathbb{A}$ be a right-contractible augmented simplicial object in a semi-abelian category $\mathcal{A}$. Then,

$$H_n\mathbb{A} = \begin{cases} A_{-1} & \text{if } n = 0 \\ 0 & \text{if } n \neq 0. \end{cases}$$

*Proof.* There is nothing to prove if $n < 0$. If $n = 0$, we know that

$$\begin{tikzcd}
A_1 \arrow[r, shift left, "\partial_0"] \arrow[r, shift right, "\partial_1"'] & A_0 \arrow[r, "\partial_0"] \arrow[l, bend left=60, "h_0"] & A_{-1} \arrow[l, bend left=60, "h_{-1}"]
\end{tikzcd}$$

is a split coequalizer diagram. So, by proposition 3.6, $H_0\mathbb{A} = A_{-1}$.

Now, if $n > 0$, we have to prove that the sequence $N(\mathbb{A})$ is exact at $N_n\mathbb{A}$, i.e. $\operatorname{im} d_{n+1} = \ker d_n$. Let $A_n^{n+1} \xrightarrow{\overline{\partial_n}} A_{n-1}^{n+1}$ and $A_{n-1}^{n+1} \xrightarrow{\overline{h_{n-1}}} A_n^{n+1}$ be the maps such that $\pi_i'\overline{\partial_n} = \partial_n\pi_i$ and $\pi_i\overline{h_{n-1}} = h_{n-1}\pi_i'$ for all $i \in [n]$. Recall that $\partial_n\partial_i = \partial_i\partial_{n+1}$ and $\partial_i h_n = h_{n-1}\partial_i$ for all $i \in [n]$. Therefore, in the following diagram, the two upward and the two downward squares commute,

$$\begin{tikzcd}
N_{n+1} \arrow[r, "\ker(\partial_j)_{j\in[n]}"] \arrow[d, shift right, "x_n"'] & A_{n+1} \arrow[r, "(\partial_j)_{j\in[n]}"] \arrow[d, shift right, "\partial_{n+1}"'] & A_n^{n+1} \arrow[d, shift right, "\overline{\partial_n}"'] \\
\operatorname{Ker} d_n \arrow[r, "\ker(\partial_j)_{j\in[n]}"'] \arrow[u, shift right, "y_n"'] & A_n \arrow[r, "(\partial_j)_{j\in[n]}"'] \arrow[u, shift right, "h_n"'] & A_{n-1}^{n+1} \arrow[u, shift right, "\overline{h_{n-1}}"']
\end{tikzcd}$$

where $x_n$ and $y_n$ and the induced morphisms. Thus, $x_ny_n = 1_{\operatorname{Ker} d_n}$ since

$$\begin{aligned}
\ker(\partial_j)_{j\in[n]} \circ x_ny_n &= \partial_{n+1} \circ \ker(\partial_j)_{j\in[n]} \circ y_n \\
&= \partial_{n+1}h_n \circ \ker(\partial_j)_{j\in[n]} \\
&= \ker(\partial_j)_{j\in[n]}.
\end{aligned}$$

So $x_n$ is a regular epimorphism. But, by lemma 3.7 and definition of $d_{n+1}$,

$$\begin{aligned}
\ker(\partial_j)_{j\in[n-1]} \circ \ker d_n \circ x_n &= \ker(\partial_j)_{j\in[n]} \circ x_n \\
&= \partial_{n+1} \circ \ker(\partial_j)_{j\in[n]} \\
&= \ker(\partial_j)_{j\in[n-1]} \circ d_{n+1}.
\end{aligned}$$

Therefore $d_{n+1} = \ker d_n \circ x_n$ which is its image factorisation. So $\operatorname{im} d_{n+1} = \ker d_n$.

□

## 3.2 Induced simplicial object and related results

It is often useful to know when two semi-simplicial maps have the same image under $H_n$. Our aim in this section is to prove a result which gives a sufficient condition for that. The key point in proving it is to construct a new simplicial object from a given one. To do so, we need to assume that the category has finite limits. In particular, it will work in our context of semi-abelian categories. But, first, let's prove some lemmas used in this proof.

**Lemma 3.9.** Let $\mathcal{A}$ be a semi-abelian category, $n \in \mathbb{N}$ and $f_i \in \mathcal{A}(B, C)$ for $i \in [n+1]$. Suppose we also have three morphisms $\begin{tikzcd} A \arrow[r, shift left=2, "x"] \arrow[r, "y" description] \arrow[r, shift right=2, "z"'] & B \end{tikzcd}$ such that $f_ix = f_iy$ for all $i \in [n]$ and $f_{n+1}y = f_{n+1}z$. Then, there exists a regular epimorphism $Z \xrightarrow{p} A$ and a map $Z \xrightarrow{g} B$ satisfying $f_ig = f_izp$ for all $i \in [n]$ and $f_{n+1}g = f_{n+1}xp$.

*Proof.* Let's consider the following diagram where all squares are pullbacks.

$$\begin{tikzcd}
R \arrow[r, "r2"] \arrow[d, "r_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & Q \arrow[r, "q_2"] \arrow[d, "q_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "f_{n+1}"] \\
P \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[r, "f_{n+1}"] \arrow[d, "(f_i)_{i\in[n]}"] & C \\
B \arrow[r, "(f_i)_{i\in[n]}"'] & C^{n+1} &
\end{tikzcd}$$

Thus, by assumption, there exist morphisms $e'$ and $e''$ making

$$\begin{tikzcd}
A \arrow[dr, "e'"] \arrow[drr, bend left, "y"] \arrow[ddr, bend right, "x"'] & & \\
& P \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "(f_i)_{i\in[n]}"] \\
& B \arrow[r, "(f_i)_{i\in[n]}"'] & C^{n+1}
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
A \arrow[dr, "e''"] \arrow[drr, bend left, "z"] \arrow[ddr, bend right, "y"'] & & \\
& Q \arrow[r, "q_2"] \arrow[d, "q_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "f_{n+1}"] \\
& B \arrow[r, "f_{n+1}"'] & C
\end{tikzcd}$$

commute. Moreover, we have a morphism $e$ for which

$$\begin{tikzcd}
A \arrow[dr, "e"] \arrow[drr, bend left, "e''"] \arrow[ddr, bend right, "e'"'] & & \\
& R \arrow[r, "r2"] \arrow[d, "r_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & Q \arrow[d, "q_1"] \\
& P \arrow[r, "p_2"'] & B
\end{tikzcd}$$

commutes. Hence, $A \xrightarrow{(x,z)} B^2 \;=\; A \xrightarrow{e} R \xrightarrow{(p_1r_1,q_2r_2)} B^2$ .

Now, let us consider one more pullback.

$$\begin{tikzcd}
S \arrow[r, "s_2"] \arrow[d, "s_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & P \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "(f_i)_{i\in[n]}"] \\
Q \arrow[r, "q_2"] \arrow[d, "q_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[r, "(f_i)_{i\in[n]}"'] \arrow[d, "f_{n+1}"] & C^{n+1} \\
B \arrow[r, "f_{n+1}"'] & C &
\end{tikzcd}$$

Let $S \xrightarrow{p'} I \xrightarrow{(i_1,i_2)} B^2$ be the image factorisation of $(q_1s_1, p_2s_2)$. Since $P \underset{p_2}{\overset{p_1}{\rightrightarrows}} B$ and $Q \underset{q_2}{\overset{q_1}{\rightrightarrows}} B$ are (effective) equivalence relations, we can find $P \xrightarrow{\sigma_P} P$ , $B \xrightarrow{t_P} P$ , $Q \xrightarrow{\sigma_Q} Q$ and $B \xrightarrow{t_Q} Q$ such that $p_1\sigma_P = p_2$, $p_2\sigma_P = p_1$, $p_1t_P = p_2t_P = 1_B$, $q_1\sigma_Q = q_2$,

$q_2\sigma_Q = q_1$ and $q_1 t_Q = q_2 t_Q = 1_B$. In particular, we can find a morphism $t_S$ such that

$$\begin{tikzcd}
B \arrow[dr, "t_S"] \arrow[drr, bend left, "t_P"] \arrow[ddr, bend right, "t_Q"'] & & \\
& S \arrow[r, "s_2"] \arrow[d, "s_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & P \arrow[d, "p_1"] \\
& Q \arrow[r, "q_2"'] & B
\end{tikzcd}$$

commutes. Thus $B \xrightarrow{t_S} S \xrightarrow{p'} I \xrightarrow{(i_1,i_2)} B^2 = B \xrightarrow{(1_B,1_B)} B^2$ and $I \overset{i_1}{\underset{i_2}{\rightrightarrows}} B$ is a reflexive relation. But since $\mathcal{A}$ is semi-abelian, it is Mal'cev (proposition 2.26) and so this relation is an equivalence. So, let $I \xrightarrow{\sigma} I$ be a map such that $i_1\sigma = i_2$ and $i_2\sigma = i_1$. In addition, we know that there is a morphism $k$ making the following diagram commute,

$$\begin{tikzcd}
R \arrow[dr, "k"] \arrow[drr, bend left, "\sigma_P r_1"] \arrow[ddr, bend right, "\sigma_Q r_2"'] & & \\
& S \arrow[r, "s_2"] \arrow[d, "s_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & P \arrow[d, "p_1"] \\
& Q \arrow[r, "q_2"'] & B
\end{tikzcd}$$

since $p_1\sigma_P r_1 = p_2 r_1 = q_1 r_2 = q_2\sigma_Q r_2$. Thus

$$\begin{aligned}
(i_1, i_2)\sigma p' k e &= (i_2, i_1)p' k e \\
&= (p_2 s_2, q_1 s_1)ke \\
&= (p_2\sigma_P r_1, q_1\sigma_Q r_2)e \\
&= (p_1 r_1, q_2 r_2)e \\
&= (x, z).
\end{aligned}$$

Finally, let us consider a last pullback

$$\begin{tikzcd}
Z \arrow[r, "p"] \arrow[d, "g'"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "\sigma p' k e"] \arrow[dr, bend left, "(x{,}z)"] & \\
S \arrow[r, "p'"] \arrow[rr, bend right, "(q_1 s_1{,}p_2 s_2)"'] & I \arrow[r, "(i_1{,}i_2)"] & B^2
\end{tikzcd}$$

where $p$ is a regular epimorphism since $p'$ is. Let $g = q_2 s_1 g'$. If $i \in [n]$, then

$$\begin{aligned}
f_i g &= f_i q_2 s_1 g' \\
&= f_i p_1 s_2 g' \\
&= f_i p_2 s_2 g' \\
&= f_i z p
\end{aligned}$$

whereas

$$\begin{aligned}
f_{n+1} g &= f_{n+1} q_2 s_1 g' \\
&= f_{n+1} q_1 s_1 g' \\
&= f_{n+1} x p.
\end{aligned}$$

Therefore, $p$ and $g$ satisfy the desired properties.

$\square$

For the next lemma, we need to introduce the notion of $X$-horns. We define here some other related concepts which will be used in the next section.

**Definition 3.10.** Let $\mathbb{A}$ be a simplicial object in a category $\mathcal{A}$. If $n \in \mathbb{N}_0$, $k \in [n]$ and $X \in \operatorname{ob} \mathcal{A}$, a **(n,k)-X-horn** is the data of $n$ maps $X \xrightarrow{x_i} A_{n-1}$ , $i \in [n] \setminus \{k\}$, such that $\partial_i x_j = \partial_{j-1} x_i$ for all $0 \leqslant i < j \leqslant n$ and $i, j \neq k$.

A **filler** for this $(n,k)$-$X$-horn is a map $X \xrightarrow{x} A_n$ such that $\partial_i x = x_i$ for all $i \in [n] \setminus \{k\}$.

We say that $\mathbb{A}$ is **X-Kan** if each $X$-horn has a filler.

**Lemma 3.11.** Let $\mathbb{A}$ be a simplicial object in a semi-abelian category $\mathcal{A}$. If $X \xrightarrow{x_i} A_{n-1}$ is a $(n,k)$-$X$-horn, then there is a regular epimorphism $Z \xrightarrow{p} X$ and a map $Z \xrightarrow{g} A_n$ satisfying $\partial_i g = x_i p$ for all $i \in [n] \setminus \{k\}$.

*Proof.* First of all, consider the case where $n = 1$. Here, we only have to consider the pullback,

$$\begin{tikzcd}
Z \arrow[r, "p"] \arrow[d, "g"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "x_{1-k}"] \\
A_1 \arrow[r, "\partial_{1-k}"'] & A_0
\end{tikzcd}$$

where $p$ and $g$ trivially satisfy the required conditions since pullbacks preserve regular epimorphisms and $\partial_i$ are split epimorphisms.

Now, we can assume that $n \geqslant 2$. If $k < n$, let's prove by downward induction on $r \in \{k+1, \ldots, n\}$, that there exists a regular epimorphism $Z_r \xrightarrow{p_r} X$ and a map $Z_r \xrightarrow{g_r} A_n$ such that $\partial_i g_r = x_i p_r$ for all $r \leqslant i \leqslant n$.

For $r = n$, set $p_n = 1_X$ and $g_n = \sigma_{n-1} x_n$. They satisfy the only desired condition since $\partial_n \sigma_{n-1} = 1_{A_{n-1}}$.

Suppose that $p_{r+1}$ and $g_{r+1}$ are constructed for $n > r \geqslant k+1$ and let's construct $p_r$ and $g_r$. We want to use our lemma 3.9 with the morphisms $\begin{tikzcd} Z_{r+1} \arrow[r, shift left=2, "\sigma_{r-1} x_r p_{r+1}"] \arrow[r, "\sigma_{r-1}\partial_r g_{r+1}" description] \arrow[r, shift right=2, "g_{r+1}"'] & A_n \end{tikzcd}$ .

If $i \in \{r+1, \ldots, n\}$, we can compute, using the simplicial identities and the definition of a $X$-horn,

$$\begin{aligned}
\partial_i \sigma_{r-1} x_r p_{r+1} &= \sigma_{r-1} \partial_{i-1} x_r p_{r+1} \\
&= \sigma_{r-1} \partial_r x_i p_{r+1} \\
&= \sigma_{r-1} \partial_r \partial_i g_{r+1} \\
&= \sigma_{r-1} \partial_{i-1} \partial_r g_{r+1} \\
&= \partial_i \sigma_{r-1} \partial_r g_{r+1}
\end{aligned}$$

whereas,

$$\partial_r \sigma_{r-1} \partial_r g_{r+1} = \partial_r g_{r+1}.$$

So, by lemma 3.9, we have a regular epimorphism $Z_r \xrightarrow{p'} Z_{r+1}$ and a map $Z_r \xrightarrow{g_r} A_n$ such that $\partial_i g_r = \partial_i g_{r+1} p'$ for all $i \in \{r+1, \ldots, n\}$ and $\partial_r g_r = \partial_r \sigma_{r-1} x_r p_{r+1} p'$. Setting $p_r = p_{r+1} p'$, we deduce $\partial_i g_r = x_i p_r$ for all $i \in \{r, \ldots, n\}$, which concludes the induction.

If $k = 0$, we can set $p = p_1$ and $g = g_1$ and the proof is finished. So, now, we can assume that $0 < k \leqslant n$. We are going to extend the previous induction: let's prove by (upward) induction on $r \in [k-1]$ that there exists a regular epimorphism $Z_r \xrightarrow{p_r} X$ and a map $Z_r \xrightarrow{g_r} A_n$ such that $\partial_i g_r = x_i p_r$ for all $0 \leqslant i \leqslant r$ and $k+1 \leqslant i \leqslant n$. For $r = 0$, we have to split the proof in two cases ($k = n$ or $k < n$), because we have not proved anything yet if $k = n$. So, if $k = n$, we can set $p_0 = 1_X$ and $g_0 = \sigma_0 x_0$ which satisfy the only required condition. Now, if $k < n$, we shall use our lemma 3.9 with

$$\begin{tikzcd} Z_{k+1} \arrow[r, shift left=2, "\sigma_0 x_0 p_{k+1}"] \arrow[r, "\sigma_0 \partial_0 g_{k+1}" description] \arrow[r, shift right=2, "g_{k+1}"'] & A_n \end{tikzcd}.$$

If $k+1 \leqslant i \leqslant n$, we can compute,

$$\begin{aligned} \partial_i \sigma_0 x_0 p_{k+1} &= \sigma_0 \partial_{i-1} x_0 p_{k+1} \\ &= \sigma_0 \partial_0 x_i p_{k+1} \\ &= \sigma_0 \partial_0 \partial_i g_{k+1} \\ &= \sigma_0 \partial_{i-1} \partial_0 g_{k+1} \\ &= \partial_i \sigma_0 \partial_0 g_{k+1} \end{aligned}$$

whereas,

$$\partial_0 \sigma_0 \partial_0 g_{k+1} = \partial_0 g_{k+1}.$$

So, by lemma 3.9, there is a regular epimorphism $Z_0 \xrightarrow{p''} Z_{k+1}$ and a map $Z_0 \xrightarrow{g_0} A_n$ such that $\partial_i g_0 = \partial_i g_{k+1} p''$ for all $k+1 \leqslant i \leqslant n$ and $\partial_0 g_0 = \partial_0 \sigma_0 x_0 p_{k+1} p''$. Hence, if we set $p_0 = p_{k+1} p''$, we have that $\partial_i g_0 = x_i p_0$ for all $k+1 \leqslant i \leqslant n$ and $i = 0$.

Now, suppose we have constructed $p_{r-1}$ and $g_{r-1}$ for $1 \leqslant r \leqslant k-1$ and let's prove the existence of $p_r$ and $g_r$. We shall use lemma 3.9 with

$$\begin{tikzcd} Z_{r-1} \arrow[r, shift left=2, "\sigma_r x_r p_{r-1}"] \arrow[r, "\sigma_r \partial_r g_{r-1}" description] \arrow[r, shift right=2, "g_{r-1}"'] & A_n \end{tikzcd}.$$

If $k+1 \leqslant i \leqslant n$,

$$\begin{aligned} \partial_i \sigma_r x_r p_{r-1} &= \sigma_r \partial_{i-1} x_r p_{r-1} \\ &= \sigma_r \partial_r x_i p_{r-1} \\ &= \sigma_r \partial_r \partial_i g_{r-1} \\ &= \sigma_r \partial_{i-1} \partial_r g_{r-1} \\ &= \partial_i \sigma_r \partial_r g_{r-1}. \end{aligned}$$

Similarly, if $0 \leqslant i \leqslant r-1$,

$$\begin{aligned} \partial_i \sigma_r x_r p_{r-1} &= \sigma_{r-1} \partial_i x_r p_{r-1} \\ &= \sigma_{r-1} \partial_{r-1} x_i p_{r-1} \\ &= \sigma_{r-1} \partial_{r-1} \partial_i g_{r-1} \\ &= \sigma_{r-1} \partial_i \partial_r g_{r-1} \\ &= \partial_i \sigma_r \partial_r g_{r-1} \end{aligned}$$

whereas,

$$\partial_r \sigma_r \partial_r g_{r-1} = \partial_r g_{r-1}.$$

Hence, again by lemma 3.9, there exists a regular epimorphism $Z_r \xrightarrow{p'''} Z_{r-1}$ and a map $Z_r \xrightarrow{g_r} A_n$ such that $\partial_i g_r = \partial_i g_{r-1} p'''$ for all $0 \leqslant i \leqslant r-1$ and $k+1 \leqslant i \leqslant n$ and $\partial_r g_r = \partial_r \sigma_r x_r p_{r-1} p'''$. Thus, if we set $p_r = p_{r-1} p'''$, we have $\partial_i g_r = x_i p_r$ for all $0 \leqslant i \leqslant r$

and $k+1 \leqslant i \leqslant n$, which concludes the second induction. Finally, we get $p$ and $g$ by setting $p = p_{k-1}$ and $g = g_{k-1}$.

$\square$

As announced earlier, here is the key point in our proof, the construction of $\mathbb{A}^I$.

**Proposition 3.12.** Let $\mathcal{A}$ be a finitely complete category and $\mathbb{A}$ a simplicial object in $\mathcal{A}$. Let us write $A^I_0 = A_1$ and let, for $n \in \mathbb{N}_0$, $A^I_n$ be the limit of

$$\begin{tikzcd}
A_{n+1} \arrow[dr, "\partial_1"] & & A_{n+1} \arrow[dl, "\partial_1"'] \arrow[dr, "\partial_2"] & & \cdots \arrow[dl, dotted] \arrow[dr, dotted] & & A_{n+1} \arrow[dl, "\partial_n"'] \\
& A_n & & A_n & & A_n &
\end{tikzcd} \tag{3.5}$$

with $\mathrm{pr}_1, \ldots, \mathrm{pr}_{n+1} \in \mathcal{A}(A^I_n, A_{n+1})$, the corresponding projections.

Moreover, we can define $A^I_n \xrightarrow{\partial^I_i} A^I_{n-1}$ for $n \in \mathbb{N}_0$ and $i \in [n]$ by

$$\partial^I_0 = \; A^I_1 \xrightarrow{\partial_0 \mathrm{pr}_2} A^I_0 \quad \text{and} \quad \partial^I_1 = \; A^I_1 \xrightarrow{\partial_2 \mathrm{pr}_1} A^I_0$$

for $n = 0$, and for $n > 0$, $\partial^I_i$ will be the map induced by

$$\mathrm{pr}_j \partial^I_i = \begin{cases} A^I_n \xrightarrow{\partial_{i+1} \mathrm{pr}_j} A_n & \text{if } 1 \leqslant j \leqslant i \\ A^I_n \xrightarrow{\partial_i \mathrm{pr}_{j+1}} A_n & \text{if } n \geqslant j > i. \end{cases}$$

Similarly, we can also define $A^I_n \xrightarrow{\sigma^I_i} A^I_{n+1}$ for $n \in \mathbb{N}$ and $i \in [n]$ to be the maps induced by

$$\mathrm{pr}_1 \sigma^I_0 = \; A^I_0 \xrightarrow{\sigma_1} A_2 \quad \text{and} \quad \mathrm{pr}_2 \sigma^I_0 = \; A^I_0 \xrightarrow{\sigma_0} A_2$$

for $n = 0$, and for $n > 0$,

$$\mathrm{pr}_j \sigma^I_i = \begin{cases} A^I_n \xrightarrow{\sigma_{i+1} \mathrm{pr}_j} A_{n+2} & \text{if } 1 \leqslant j \leqslant i+1 \\ A^I_n \xrightarrow{\sigma_i \mathrm{pr}_{j-1}} A_{n+2} & \text{if } n+2 \geqslant j > i+1. \end{cases}$$

These constructions make $\mathbb{A}^I$ be a simplicial object in $\mathcal{A}$.

Finally, we can define simplicial maps $\epsilon_0(\mathbb{A}), \epsilon_1(\mathbb{A}) \in \mathrm{S}\,\mathcal{A}(\mathbb{A}^I, \mathbb{A})$ and $s(\mathbb{A}) \in \mathrm{S}\,\mathcal{A}(\mathbb{A}, \mathbb{A}^I)$ by

$$\epsilon_0(\mathbb{A})_0 = \; A^I_0 \xrightarrow{\partial_0} A_0 \quad \text{and} \quad \epsilon_0(\mathbb{A})_n = \; A^I_n \xrightarrow{\partial_0 \mathrm{pr}_1} A_n \quad \text{for } n > 0,$$

$$\epsilon_1(\mathbb{A})_0 = \; A^I_0 \xrightarrow{\partial_1} A_0 \quad \text{and} \quad \epsilon_1(\mathbb{A})_n = \; A^I_n \xrightarrow{\partial_{n+1} \mathrm{pr}_{n+1}} A_n \quad \text{for } n > 0$$

and

$$s(\mathbb{A})_0 = \; A_0 \xrightarrow{\sigma_0} A^I_0 \quad \text{and} \quad \mathrm{pr}_j s(\mathbb{A})_n = \sigma_{j-1} \quad \text{for } 1 \leqslant j \leqslant n+1 \text{ and } n > 0.$$

These maps satisfy $\epsilon_0(\mathbb{A}) s(\mathbb{A}) = \epsilon_1(\mathbb{A}) s(\mathbb{A}) = 1_{\mathbb{A}}$.

*Proof.* First of all, we have to check that $\partial_i^I$ is well defined for all $n \in \mathbb{N}_0$, i.e. that $\partial_j \mathrm{pr}_j \partial_i^I = \partial_j \mathrm{pr}_{j+1} \partial_i^I$ for all $j \in \{1, \dots, n-1\}$. We also have to do it for $\sigma_i^I$ with $n \in \mathbb{N}$.

Then, we proof that $\mathbb{A}^I$ is a simplicial object, i.e. that the simplicial identities (3.1), (3.2) and (3.3) are satisfied by $\partial_i^I$ and $\sigma_i^I$.

Afterwards, we have to check that $s(\mathbb{A})_n$ is well defined for all $n \in \mathbb{N}_0$ and that $\epsilon_0(\mathbb{A})$, $\epsilon_1(\mathbb{A})$ and $s(\mathbb{A})$ are simplicial maps, i.e. that they commute appropriately with the face and degeneracy operators.

Finally, we have to check that $\epsilon_0(\mathbb{A})_n s(\mathbb{A})_n = \epsilon_1(\mathbb{A})_n s(\mathbb{A})_n = 1_{\mathbb{A}_n}$ for all $n \in \mathbb{N}$.

To prove all these things, we only have to use the definitions of simplicial object, $A_n^I$ and the different maps used. This is very long, but there is not any difficulty, that is why we omitted the details.

□

Now we are going to prove that, in a semi-abelian category, $H_n N(\epsilon_0(\mathbb{A}))$ is actually an isomorphism. Recall that, by abuse of notation, we denote it by $H_n \epsilon_0(\mathbb{A})$ (see definition 3.5). Our sufficient condition for two semi-simplicial maps to have equal image under $H_n N$ will trivially follow from this.

**Proposition 3.13.** Let $\mathbb{A}$ be a simplicial object in a semi-abelian category $\mathcal{A}$. Consider the map $\epsilon_0(\mathbb{A}) \in \mathrm{S}\,\mathcal{A}(\mathbb{A}^I, \mathbb{A})$ from proposition 3.12. Then, $H_n \epsilon_0(\mathbb{A})$ is an isomorphism for all $n \in \mathbb{Z}$.

*Proof.* Since for $n < 0$ there is nothing to prove, we can assume $n \in \mathbb{N}$. Recall that $Z_n \epsilon_0(\mathbb{A})$ is the morphism such that $\ker d_n \circ Z_n \epsilon_0(\mathbb{A}) = N_n \epsilon_0(\mathbb{A}) \circ \ker d_n^I$. Thus $Z_n \epsilon_0(\mathbb{A}) d'^I_{n+1} = d'_{n+1} N_{n+1} \epsilon_0(\mathbb{A})$ since they are equal when we compose them with the monomorphism $\ker d_n$.

$$\begin{tikzcd}
N_{n+1}\mathbb{A}^I \arrow[rr, "d^I_{n+1}"] \arrow[dr, "d'^I_{n+1}"'] \arrow[dd, "N_{n+1}\epsilon_0(\mathbb{A})"'] & & N_n\mathbb{A}^I \arrow[r, "d^I_n"] \arrow[dd, "N_n\epsilon_0(\mathbb{A})"] & N_{n-1}\mathbb{A}^I \arrow[dd, "N_{n-1}\epsilon_0(\mathbb{A})"] \\
& \operatorname{Ker} d^I_n \arrow[ur, "\ker d^I_n"'] \arrow[dd, dotted, "Z_n\epsilon_0(\mathbb{A})", near start] & & \\
N_{n+1}\mathbb{A} \arrow[rr, "d_{n+1}"] \arrow[dr, "d'_{n+1}"'] & & N_n\mathbb{A} \arrow[r, "d_n"] & N_{n-1}\mathbb{A} \\
& \operatorname{Ker} d_n \arrow[ur, "\ker d_n"'] & &
\end{tikzcd}$$

Now, consider the following pullback with $e$ the induced morphism.

$$\begin{tikzcd}
N_{n+1}\mathbb{A}^I \arrow[dr, "e"] \arrow[drr, bend left, "N_{n+1}\epsilon_0(\mathbb{A})"] \arrow[ddr, bend right, "d'^I_{n+1}"'] & & \\
& P_0 \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, very near start, "\lrcorner"] & N_{n+1}\mathbb{A} \arrow[d, "d'_{n+1}"] \\
& \operatorname{Ker} d^I_n \arrow[r, "Z_n\epsilon_0(\mathbb{A})"'] & \operatorname{Ker} d_n
\end{tikzcd}$$

We are going to prove that $e$ is a regular epimorphism. Let $p = \ker(\partial_j)_{j \in [n]} \circ p_1$. So, by definition of the action of $N_{n+1}$ on arrows, the following diagram commute,

$$\begin{tikzcd}
N_{n+1}\mathbb{A}^I \arrow[r, "\ker\left(\partial^I_j\right)_{j\in[n]}"] \arrow[d, "N_{n+1}\epsilon_0(\mathbb{A})"'] & A^I_{n+1} \arrow[r, "\left(\partial^I_j\right)_{j\in[n]}"] \arrow[d, "\epsilon_0(\mathbb{A})_{n+1}"] & {A^I}^{n+1}_n \arrow[d, "\overline{\epsilon_0(\mathbb{A})_n}"] \\
N_{n+1}\mathbb{A} \arrow[r, "\ker(\partial_j)_{j\in[n]}"'] & A_{n+1} \arrow[r, "(\partial_j)_{j\in[n]}"'] & A^{n+1}_n \\
P_0 \arrow[u, "p_1"] \arrow[ru, bend right, "p"'] & &
\end{tikzcd}$$

where $\overline{\epsilon_0(\mathbb{A})_n}$ is the unique map such that $\pi_i \circ \overline{\epsilon_0(\mathbb{A})_n} = \epsilon_0(\mathbb{A})_n \circ \pi^I_i$ for all $i \in [n]$. Thus, by definition of $p$, $\partial_i p = 0$ for all $i \in [n]$.

By lemma 3.7, we know that if $n \geqslant 1$, $\ker\left(\partial^I_j\right)_{j\in[n-1]} \circ \ker d^I_n = \ker\left(\partial^I_j\right)_{j\in[n]}$. So, if $n \geqslant 1$, let $q_{k-1}$, for $k \in \{1,\dots,n+1\}$, be the following composite.

$$\begin{tikzcd}
P_0 \arrow[r, "p_2"] \arrow[rrrd, bend right, "q_{k-1}"'] & \operatorname{Ker} d^I_n \arrow[r, "\ker d^I_n"] \arrow[rd, "\ker\left(\partial^I_j\right)_{j\in[n]}"'] & N_n\mathbb{A}^I \arrow[d, "\ker\left(\partial^I_j\right)_{j\in[n-1]}"] & \\
 & & A^I_n \arrow[r, "\mathrm{pr}_k"] & A_{n+1}
\end{tikzcd}$$

By definition of $A^I_n$, we know that $\partial_k q_{k-1} = \partial_k q_k$ for all $1 \leqslant k \leqslant n$. Moreover, in view of definition of $\partial^I_i$, we can say that, for all $1 \leqslant k \leqslant n+1$ and $i \in [n+1]\backslash\{k-1,k\}$, $\partial_i q_{k-1} = 0$. If $n = 0$, let $q_0 = p_2 \in \mathcal{A}(P, A_1)$ since $\operatorname{Ker} d^I_0 = \mathbb{A}^I_0 = A_1$. Let's prove that $\partial_{n+1} p = \partial_0 q_0$. If $n = 0$, $\partial_1 p = \partial_1(\ker\partial_0)p_1 = d_1 p_1 = d'_1 p_1 = Z_0\epsilon_0(\mathbb{A})p_2 = N_0\epsilon_0(\mathbb{A})p_2 = \partial_0 p_2$. Otherwise,

$$\begin{aligned}
\partial_{n+1} p &= \partial_{n+1}\ker(\partial_j)_{j\in[n]}p_1 = \ker(\partial_j)_{j\in[n-1]}d_{n+1}p_1 \\
&= \ker(\partial_j)_{j\in[n-1]}\ker d_n d'_{n+1}p_1 = \ker(\partial_j)_{j\in[n-1]}\ker d_n Z_n\epsilon_0(\mathbb{A})p_2 \\
&= \ker(\partial_j)_{j\in[n-1]}N_n\epsilon_0(\mathbb{A})\ker d^I_n p_2 = \epsilon_0(\mathbb{A})_n\ker(\partial^I_j)_{j\in[n-1]}\ker d^I_n p_2 \\
&= \partial_0\mathrm{pr}_1\ker(\partial^I_j)_{j\in[n]}p_2 = \partial_0 q_0.
\end{aligned}$$

Now, we are going to construct, by induction on $r \in \{1,\dots,n+2\}$, regular epimorphisms $P_r \xrightarrow{y_r} P_{r-1}$ and maps $P_r \xrightarrow{h_{r-1}} A_{n+2}$ satisfying $\partial_i h_{r-1} = 0$ for $i \notin \{r-1, r, n+2\}$, $\partial_{n+2}h_{r-1} = q_{r-1}y_1\dots y_r$ if $r \leqslant n+1$, $\partial_0 h_0 = py_1$ and $\partial_{r-1}h_{r-2}y_r = \partial_{r-1}h_{r-1}$ if $r \geqslant 2$. For $r = 1$, we define $x_0 = p$, $x_2 = \dots x_{n+1} = 0$ and $x_{n+2} = q_0$. This is a $(n+2,1)$-$P_0$-horn since $\partial_i p = 0$ for all $i \in [n]$, $\partial_{n+1}p = \partial_0 q_0$ and $\partial_i q_0 = 0$ for all $2 \leqslant i \leqslant n+1$. So, by lemma 3.11, there is a regular epimorphism $P_1 \xrightarrow{y_1} P_0$ and a map $P_1 \xrightarrow{h_0} A_{n+2}$ such that $\partial_0 h_0 = py_1$, $\partial_i h_0 = 0$ for all $2 \leqslant i \leqslant n+1$ and $\partial_{n+2}h_0 = q_0 y_1$.

Suppose that $n \geqslant 1$ and $y_r$ and $h_{r-1}$ has been constructed for $1 \leqslant r \leqslant n$ and let's prove the existence of $y_{r+1}$ and $h_r$. Let $x_0 = \cdots = x_{r-1} = 0$, $x_r = \partial_r h_{r-1}$, $x_{r+2} = \cdots = x_{n+1} = 0$ and $x_{n+2} = q_r y_1 \dots y_r$. We know that $\partial_{j-1}\partial_r h_{r-1} = \partial_r\partial_j h_{r-1} = 0$ for all $j \in \{r+2,\dots,n+1\}$ and $\partial_r q_r y_1\dots y_r = \partial_r q_{r-1}y_1\dots y_r = \partial_r\partial_{n+2}h_{r-1} = \partial_{n+1}\partial_r h_{r-1}$. Moreover, if $i < r-1$, $\partial_i\partial_r h_{r-1} = \partial_{r-1}\partial_i h_{r-1} = 0$, whereas, if $i = r-1$:

$$\partial_{r-1}\partial_r h_{r-1} = \partial_{r-1}\partial_{r-1}h_{r-1} = \partial_{r-1}\partial_{r-1}h_{r-2}y_r = \partial_{r-1}\partial_r h_{r-2}y_r = 0 \quad \text{if } r > 1$$

and

$$\partial_0\partial_1 h_0 = \partial_0\partial_0 h_0 = \partial_0 p y_1 = 0 \quad \text{if } r = 1.$$

Finally, $\partial_i q_r y_1 \dots y_r = 0$ if $i \notin \{r, r+1, n+2\}$. Therefore, the $x_i$'s form a $(n+2, r+1)$-$P_r$-horn. So, by lemma 3.11, there exists a regular epimorphism $P_{r+1} \xrightarrow{y_{r+1}} P_r$ and a map $P_{r+1} \xrightarrow{h_r} A_{n+2}$ such that $\partial_i h_r = 0$ for all $i \notin \{r, r+1, n+2\}$, $\partial_r h_r = \partial_r h_{r-1} y_{r+1}$ and $\partial_{n+2} h_r = q_r y_1 \dots y_r y_{r+1}$.

It remains to complete the last part of the induction. So, suppose all the $y_r$'s and $h_{r-1}$'s are constructed but $y_{n+2}$ and $h_{n+1}$ and let's prove their existence. We define $x_0 = \dots = x_n = 0$ and $x_{n+1} = \partial_{n+1} h_n$. We know that $\partial_i \partial_{n+1} h_n = \partial_n \partial_i h_n = 0$ if $i < n$, whereas, if $i = n$,

$$\partial_n \partial_{n+1} h_n = \partial_n \partial_n h_n = \partial_n \partial_n h_{n-1} y_{n+1} = \partial_n \partial_{n+1} h_{n-1} y_{n+1} = 0 \quad \text{if } n > 0$$

and $\partial_0 \partial_0 h_0 = \partial_0 p y_1 = 0$ if $n = 0$. So the $x_i$'s form a $(n+2, n+2)$-$P_{n+1}$-horn. Hence, by lemma 3.11, there exists a regular epimorphism $P_{n+2} \xrightarrow{y_{n+2}} P_{n+1}$ and a map $P_{n+2} \xrightarrow{h_{n+1}} A_{n+2}$ such that $\partial_i h_{n+1} = 0$ for all $i \in [n]$ and $\partial_{n+1} h_{n+1} = \partial_{n+1} h_n y_{n+2}$, which concludes the induction.

The map $h' = (h_0 y_2 \dots y_{n+2}, \dots, h_n y_{n+2}, h_{n+1}) \in \mathcal{A}(P_{n+2}, A^I_{n+1})$ is well defined, since $\partial_i h_{i-1} y_{i+1} = \partial_i h_i$ if $1 \leqslant i \leqslant n+1$. We would like to show $\left(\partial^I_j\right)_{j \in [n]} h' = 0$. If $n = 0$, we have $\partial^I_0 h' = \partial_0 h_1 = 0$. For $n > 0$, it is enough to show that for all $i \in [n]$ and $j \in \{1, \dots, n+1\}$, we have $\mathrm{pr}_j \partial^I_i h' = 0$. If $j \leqslant i$, $\mathrm{pr}_j \partial^I_i h' = \partial_{i+1} \mathrm{pr}_j h' = \partial_{i+1} h_{j-1} y_{j+1} \dots y_{n+2} = 0$. If $i < j < n+1$, $\mathrm{pr}_j \partial^I_i h' = \partial_i \mathrm{pr}_{j+1} h' = \partial_i h_j y_{j+2} \dots y_{n+2} = 0$, whereas, if $j = n+1$, $\mathrm{pr}_{n+1} \partial^I_i h' = \partial_i \mathrm{pr}_{n+2} h' = \partial_i h_{n+1} = 0$. Therefore, there is a map $P_{n+2} \xrightarrow{h} N_{n+1} \mathbb{A}^I$ such that $\ker \left(\partial^I_j\right)_{j \in [n]} \circ h = h'$.

Now, we would like to show that $eh = y_1 \dots y_{n+2}$. To do so, we have to prove that $p_1 e h = p_1 y_1 \dots y_{n+2}$ and $p_2 e h = p_2 y_1 \dots y_{n+2}$. For the first equality, we have to show $N_{n+1} \epsilon_0(\mathbb{A}) h = p_1 y_1 \dots y_{n+2}$, which is a consequence of

$$\begin{aligned} \ker(\partial_j)_{j \in [n]} N_{n+1} \epsilon_0(\mathbb{A}) h &= \epsilon_0(\mathbb{A})_{n+1} \ker \left(\partial^I_j\right)_{j \in [n]} h \\ &= \epsilon_0(\mathbb{A})_{n+1} h' \\ &= \partial_0 h_0 y_2 \dots y_{n+2} \\ &= p y_1 \dots y_{n+2} \\ &= \ker(\partial_j)_{j \in [n]} p_1 y_1 \dots y_{n+2}. \end{aligned}$$

For the second one, we have to prove $d'^I_{n+1} h = p_2 y_1 \dots y_{n+2}$, or $d^I_{n+1} h = \ker d^I_n p_2 y_1 \dots y_{n+2}$. If $n = 0$, we have $d^I_1 h = \partial^I_1 \ker(\partial^I_0) h = \partial_2 \mathrm{pr}_1 h' = \partial_2 h_0 y_2 = q_0 y_1 y_2 = p_2 y_1 y_2$. Otherwise, it is enough to prove $\ker \left(\partial^I_j\right)_{j \in [n-1]} d^I_{n+1} h = \ker \left(\partial^I_j\right)_{j \in [n]} p_2 y_1 \dots y_{n+2}$. Let's prove that their projections on $A_{n+1}$ are equal. So, if $k \in \{1, \dots n+1\}$, we have

$$\begin{aligned} \mathrm{pr}_k \ker \left(\partial^I_j\right)_{j \in [n-1]} d^I_{n+1} h &= \mathrm{pr}_k \partial^I_{n+1} \ker \left(\partial^I_j\right)_{j \in [n]} h \\ &= \partial_{n+2} \mathrm{pr}_k h' \\ &= \partial_{n+2} h_{k-1} y_{k+1} \dots y_{n+2} \\ &= q_{k-1} y_1 \dots y_{n+2} \\ &= \mathrm{pr}_k \ker \left(\partial^I_j\right)_{j \in [n]} p_2 y_1 \dots y_{n+2}. \end{aligned}$$

Therefore $eh = y_1 \dots y_{n+2}$ and because the $y_i$'s are regular epimorphisms, by corollary 2.23, $e$ is a regular epimorphism.

Now, consider the following diagram

$$\begin{tikzcd}
\operatorname{Ker} N_{n+1}\epsilon_0(\mathbb{A}) \arrow[rr, "\ker N_{n+1}\epsilon_0(\mathbb{A})"] \arrow[dr, "k_1"] \arrow[dd, "k_3"'] & & N_{n+1}\mathbb{A}^I \arrow[rr, "N_{n+1}\epsilon_0(\mathbb{A})"] \arrow[dd, "d'^I_{n+1}"'] \arrow[dr, "e"] & & N_{n+1}\mathbb{A} \arrow[dd, "d'_{n+1}"'] \arrow[dr, equal] & \\
& \operatorname{Ker} p_1 \arrow[rr, "\ker p_1"'] \arrow[dl, "k_2"] & & P_0 \arrow[rr, "p_1"'] \arrow[dl, "p_2"] \arrow[dr, phantom, "\lrcorner", very near start] & & N_{n+1}\mathbb{A} \arrow[dl, "d'_{n+1}"] \\
\operatorname{Ker} Z_n\epsilon_0(\mathbb{A}) \arrow[rr, "\ker Z_n\epsilon_0(\mathbb{A})"'] & & \operatorname{Ker} d^I_n \arrow[rr, "Z_n\epsilon_0(\mathbb{A})"'] & & \operatorname{Ker} d_n &
\end{tikzcd}$$

where $k_1$, $k_2$ and $k_3$ are the induced morphisms. Notice that $k_2k_1 = k_3$ since their composite with $\ker Z_n\epsilon_0(\mathbb{A})$ are equal. We also know that $\epsilon_0(\mathbb{A})$ is a split epimorphism, so $N_{n+1}\epsilon_0(\mathbb{A})$, $Z_n\epsilon_0(\mathbb{A})$ and $p_1$ are regular epimorphisms. Thus, by a well-known result about kernel and pullbacks (see lemma 4.2.4 in [4]), $k_2$ is an isomorphism. Moreover, again by lemma 4.2.4 in [4], the up-left-hand square is a pullback. So, $k_1$, and therefore $k_3$, is a regular epimorphism.

Let's prove that

$$\begin{tikzcd}
N_{n+1}\mathbb{A}^I \arrow[r, "N_{n+1}\epsilon_0(\mathbb{A})"] \arrow[d, "d'^I_{n+1}"'] & N_{n+1}\mathbb{A} \arrow[d, "d'_{n+1}"] \\
\operatorname{Ker} d^I_n \arrow[r, "Z_n\epsilon_0(\mathbb{A})"'] & \operatorname{Ker} d_n
\end{tikzcd}$$

is a pushout. Suppose we have two arrows $N_{n+1}\mathbb{A} \xrightarrow{f_1} B$ and $\operatorname{Ker} d^I_n \xrightarrow{f_2} B$ such that $f_1N_{n+1}\epsilon_0(\mathbb{A}) = f_2d'^I_{n+1}$. So, because $f_2 \ker Z_n\epsilon_0(\mathbb{A})k_3 = f_1N_{n+1}\epsilon_0(\mathbb{A}) \ker N_{n+1}\epsilon_0(\mathbb{A}) = 0$, we know that $f_2 \ker Z_n\epsilon_0(\mathbb{A}) = 0$. But $Z_n\epsilon_0(\mathbb{A})$ is a normal epimorphism, so $Z_n\epsilon_0(\mathbb{A}) = \operatorname{coker}(\ker Z_n\epsilon_0(\mathbb{A}))$. Consequently, there is a unique morphism $\operatorname{Ker} d_n \xrightarrow{g} B$ such that $gZ_n\epsilon_0(\mathbb{A}) = f_2$. This map also satisfies $gd'_{n+1} = f_1$ since their composite with $N_{n+1}\epsilon_0(\mathbb{A})$ are equal. Therefore, we have proved that the left-hand square is a pushout.

$$\begin{tikzcd}
N_{n+1}\mathbb{A}^I \arrow[r, "d'^I_{n+1}"] \arrow[d, "N_{n+1}\epsilon_0(\mathbb{A})"'] & \operatorname{Ker} d^I_n \arrow[r, "\operatorname{coker} d'^I_{n+1}"] \arrow[d, "Z_n\epsilon_0(\mathbb{A})"] & H_n\mathbb{A}^I \arrow[d, "H_n\epsilon_0(\mathbb{A})"] \\
N_{n+1}\mathbb{A} \arrow[r, "d'_{n+1}"'] \arrow[ur, phantom, "\ulcorner", very near end] & \operatorname{Ker} d_n \arrow[r, "\operatorname{coker} d'_{n+1}"'] & H_n\mathbb{A}
\end{tikzcd}$$

Therefore, by the dual of lemma 4.2.4 in [4], $H_n\epsilon_0(\mathbb{A})$ is an isomorphism.

□

Finally, we have our main result of this section as an immediate corollary of the previous proposition. Recall that, by abuse of notation, the functor $H_n$ means here $H_nN$.

**Corollary 3.14.** Let $\mathbb{A}$ and $\mathbb{B}$ be two simplicial objects in a semi-abelian category $\mathcal{A}$, with $\mathbb{A} \xrightarrow{f,g} \mathbb{B}$ two semi-simplicial maps. If there is a semi-simplicial map $\mathbb{A} \xrightarrow{h} \mathbb{B}^I$ such that $f = \epsilon_0(\mathbb{B})h$ and $g = \epsilon_1(\mathbb{B})h$, then $H_nf = H_ng$ for all $n \in \mathbb{Z}$.

*Proof.* By proposition 3.12, we know that the following diagram commutes.

$$\begin{tikzcd}
 & & \mathbb{B} \arrow[d, equal] \\
\mathbb{A} \arrow[r, "h"] \arrow[rru, bend left, "f"] \arrow[rrd, bend right, "g"'] & \mathbb{B}^I \arrow[ru, "\epsilon_0(\mathbb{B})"] \arrow[rd, "\epsilon_1(\mathbb{B})"'] & \mathbb{B} \arrow[l, "s(\mathbb{B})"'] \arrow[d, equal] \\
 & & \mathbb{B}
\end{tikzcd}$$

Moreover, by proposition 3.13, $H_n\epsilon_0(\mathbb{B})^{-1} = H_n s(\mathbb{B}) = H_n\epsilon_1(\mathbb{B})^{-1}$. Therefore, $H_n f = H_n\epsilon_0(\mathbb{B})H_n h = H_n\epsilon_1(\mathbb{B})H_n h = H_n g$.

□

## 3.3 Comparison theorem

A natural question that one can wonder about augmented simplicial objects is the following: Given two augmented simplicial objects $\mathbb{A}$ and $\mathbb{B}$ and a map $A_{-1} \xrightarrow{f_{-1}} B_{-1}$, can we extend it to a augmented simplicial map from $\mathbb{A}$ to $\mathbb{B}$? Is such an extension unique? The aim of this section is to give a sufficient condition for the existence of a semi-simplicial extension. Moreover, we shall prove that if this condition is satisfied, two such extensions have the same image under $H_n$, using corollary 3.14. To find this condition, we have to introduce the concept of a projective class.

**Definition 3.15.** Let $\mathcal{C}$ be a category, $P \in \mathrm{ob}\mathcal{C}$ and $e \in \mathcal{C}(A, B)$. We say that $P$ is **e-projective** and $e$ is **P-epic** if for any morphism $P \xrightarrow{f} B$, there exists a map $P \xrightarrow{g} A$ such that $eg = f$.

$$\begin{tikzcd}
 & P \arrow[d, "\forall f"] \arrow[ld, dotted, bend right, "\exists g"'] \\
A \arrow[r, "e"'] & B
\end{tikzcd}$$

If $\mathcal{P}$ is a class of object of $\mathcal{C}$, a morphism $e$ is called **$\mathcal{P}$-epic** if it is $P$-epic for all $P \in \mathcal{P}$. The class of morphisms which are $\mathcal{P}$-epic is denoted by **$\mathcal{P}$-epi**. Similarly, if $\mathcal{E}$ is a class of morphism of $\mathcal{C}$, a object $P$ is called **$\mathcal{E}$-projective** if it is $e$-projective for all $e \in \mathcal{E}$. The class of objects which are $\mathcal{E}$-projective is denoted by **$\mathcal{E}$-proj**.

We say that $\mathcal{C}$ **has enough $\mathcal{E}$-projectives** if for all $Z \in$ ob $\mathcal{C}$, there exists $P \in \mathcal{E}$-proj and a morphism $P \longrightarrow Z$ in $\mathcal{E}$.

Let $\mathcal{P}$ be a class of object of $\mathcal{C}$ and $\mathcal{E}$ be a class of morphisms of $\mathcal{C}$. We say that $(\mathcal{P}, \mathcal{E})$ is a **projective class on** $\mathcal{C}$ if $\mathcal{P} = \mathcal{E}$-proj, $\mathcal{E} = \mathcal{P}$-epi and $\mathcal{C}$ has enough $\mathcal{E}$-projectives. Notice that $\mathcal{P}$ and $\mathcal{E}$ determine each-other, so, by abuse of notation, we can call this projective class $\mathcal{P}$ or $\mathcal{E}$.

**Remark 3.16.** For a class $\mathcal{P} \subset$ ob $\mathcal{C}$ and a class $\mathcal{E} \subset \mathrm{mor}\mathcal{C}$, we always have $\mathcal{P} \subset$ ($\mathcal{P}$-epi)-proj and $\mathcal{E} \subset$ ($\mathcal{E}$-proj)-epi. Moreover, the pair (($\mathcal{P}$-epi)-proj, $\mathcal{P}$-epi) (respectively ($\mathcal{E}$-proj, ($\mathcal{E}$-proj)-epi)) is a projective class provided the fact that $\mathcal{C}$ has enough $\mathcal{P}$-epi-projectives (respectively ($\mathcal{E}$-proj)-epi-projectives). In that case, we call it **the projective class generated by** $\mathcal{P}$ (respectively $\mathcal{E}$).

Some definitions we have made for simplicial objects can be adapted to be relative to a projective class $\mathcal{P}$.

**Definition 3.17.** Let $\mathcal{P}$ be a projective class on a category $\mathcal{C}$ and $\mathbb{A}$ be an augmented simplicial object in $\mathcal{C}$. We say that $\mathbb{A}$ is **$\mathcal{P}$-left-contractible** if for each $P \in \mathcal{P}$, there exist mappings of classes $\mathcal{C}(P, A_n) \xrightarrow{h_n} \mathcal{C}(P, A_{n+1})$ for each $n \geqslant -1$ such that $\partial_0 \circ h_n(f) = f$ for all $f \in \mathcal{C}(P, A_n)$ and $\partial_i \circ h_n(f) = h_{n-1}(\partial_{i-1} \circ f)$ for all $f \in \mathcal{C}(P, A_n)$ and $i \in \{1, \ldots, n+1\}$.

**Remark 3.18.** Of course, we can notice similarities with the definition of left-contractible augmented simplicial objects (see definition 3.2). Actually, if $\mathcal{C}$ is locally small, $\mathbb{A}$ is $\mathcal{P}$-left-contractible if, for each $P \in \mathcal{P}$, the simplicial object $\mathcal{C}(P, \mathbb{A})$ in Set is left-contractible. The face and degeneracy operators of this simplicial object are defined obviously, i.e. $\mathcal{C}(P, A_n) \xrightarrow{\partial'_i = \partial_i \circ -} \mathcal{C}(P, A_{n-1})$ and $\mathcal{C}(P, A_n) \xrightarrow{\sigma'_i = \sigma_i \circ -} \mathcal{C}(P, A_{n+1})$ .

**Definition 3.19.** Let $\mathcal{P}$ be a projective class on a category $\mathcal{C}$ and $\mathbb{A}$ be a simplicial object in $\mathcal{C}$. We say that $\mathbb{A}$ is **$\mathcal{P}$-Kan** if it is $P$-Kan for all $P \in \mathcal{P}$.

The class $\mathcal{P}$ is a **Kan projective class on** $\mathcal{C}$ if every $\mathcal{P}$-left-contractible augmented simplicial object is $\mathcal{P}$-Kan.

Before proving the main theorem of this section, we are going to show a lemma which will be used several times in the proof of this theorem.

**Lemma 3.20.** Let $\mathcal{P}$ be a projective class on a category $\mathcal{C}$ and $\mathbb{A}$ a $\mathcal{P}$-Kan $\mathcal{P}$-left-contractible augmented simplicial object in $\mathcal{C}$. Given $P \in \mathcal{P}$, $n \in \mathbb{N}$ and $n+1$ maps $P \xrightarrow{x_i} A_{n-1}$ , $i \in [n]$, such that $\partial_i x_j = \partial_{j-1} x_i$ for all $0 \leqslant i < j \leqslant n$, then there exists a morphism $P \xrightarrow{x} A_n$ such that $\partial_i x = x_i$ for all $i \in [n]$.

*Proof.* Let $\mathcal{C}(P, A_m) \xrightarrow{h_m} \mathcal{C}(P, A_{m+1})$ , $m \geqslant -1$, be the mappings given by definition of $\mathcal{P}$-left-contractible for $P$. Define also, for $i \in \{1, \ldots, n+1\}$, $P \xrightarrow{y_i = h_{n-1}(x_{i-1})} A_n$ . By assumptions, we have (if $n > 0$), for all $1 \leqslant i < j \leqslant n+1$,

$$\begin{aligned}
\partial_i y_j &= \partial_i h_{n-1}(x_{j-1}) \\
&= h_{n-2}(\partial_{i-1} x_{j-1}) \\
&= h_{n-2}(\partial_{j-2} x_{i-1}) \\
&= \partial_{j-1} h_{n-1}(x_{i-1}) \\
&= \partial_{j-1} y_i .
\end{aligned}$$

Thus, the $y_i$'s form a $(n+1, 0)$-$P$-horn, even if $n = 0$ since there is nothing to check in that case. But $\mathbb{A}$ is $P$-Kan, so we have a filler $P \xrightarrow{y} A_{n+1}$ for this horn. So, by definition of a filler, if we set $x = \partial_0 y$, we have

$$\begin{aligned}
\partial_i x &= \partial_i \partial_0 y \\
&= \partial_0 \partial_{i+1} y \\
&= \partial_0 y_{i+1} \\
&= \partial_0 h_{n-1}(x_i) \\
&= x_i
\end{aligned}$$

for all $i \in [n]$.

□

Notice that if we have an (augmented) simplicial object $\mathbb{A}$ in a category $\mathcal{C}$ and a functor $E : \mathcal{C} \to \mathcal{A}$, then $E(\mathbb{A})$ is an (augmented) simplicial object in $\mathcal{A}$ by setting $E(\mathbb{A})_n = E(A_n)$, $\partial'_i = E(\partial_i)$ and $\sigma'_i = E(\sigma_i)$. This turns $E$ into functors $E : \mathrm{S}\,\mathcal{C} \to \mathrm{S}\,\mathcal{A}$, $\mathrm{S}'\,\mathcal{C} \to \mathrm{S}'\,\mathcal{A}$, $\mathrm{AS}\,\mathcal{C} \to \mathrm{AS}\,\mathcal{A}$ or $\mathrm{AS}'\,\mathcal{C} \to \mathrm{AS}'\,\mathcal{A}$.

Now, we are able to prove a theorem to extend morphisms $P_{-1} \longrightarrow A_{-1}$ to augmented semi-simplicial maps $\mathbb{P} \longrightarrow \mathbb{A}$. Note that we do not need the fact that $P_{-1} \in \mathcal{P}$.

**Theorem 3.21** (Comparison Theorem)**.** *Let $\mathcal{P}$ be a projective class on a category $\mathcal{C}$. Let also $\mathbb{A}$ and $\mathbb{P}$ be two augmented simplicial objects in $\mathcal{C}$ such that $P_n \in \mathcal{P}$ for all $n \in \mathbb{N}$ and $\mathbb{A}$ is $\mathcal{P}$-Kan and $\mathcal{P}$-left-contractible. Then, each morphism $P_{-1} \xrightarrow{f_{-1}} A_{-1}$ can be extended to a augmented semi-simplicial map $\mathbb{P} \xrightarrow{f} \mathbb{A}$. Moreover, if $E : \mathcal{C} \to \mathcal{A}$ is a functor to a semi-abelian category $\mathcal{A}$ and $g$ another extension, then $H_nEf = H_nEg$ for all $n \in \mathbb{Z}$.*

*Proof.* For the first part of the proof, we are going to construct the $f_n$'s by induction. Since $f_{-1}$ is already defined, let us suppose that all the $f_k$'s are constructed for $-1 \leqslant k \leqslant n-1$, $n \in \mathbb{N}$, and that they commute appropriately with the face operators. Now, we are going to construct $f_n$. Set $x_i = f_{n-1}\partial_i$ for $i \in [n]$. We want to use lemma 3.20. If $n = 0$, there is nothing to check. If $n > 0$, we can compute, for $0 \leqslant i < j \leqslant n$,

$$\begin{aligned}
\partial_i x_j &= \partial_i f_{n-1} \partial_j \\
&= f_{n-2} \partial_i \partial_j \\
&= f_{n-2} \partial_{j-1} \partial_i \\
&= \partial_{j-1} f_{n-1} \partial_i \\
&= \partial_{j-1} x_i.
\end{aligned}$$

So, lemma 3.20 gives us the expected $f_n$.

$$\begin{tikzcd}
P_n \arrow[r, shift left=4.5] \arrow[r, dotted, shift left=1.5] \arrow[r, dotted, shift right=1.5] \arrow[r, shift right=4.5] \arrow[d, dotted, "f_n"'] & P_{n-1} \arrow[r, shift left=3] \arrow[r, dotted] \arrow[r, shift right=3] \arrow[d, "f_{n-1}"] & P_{n-2} \arrow[d, "f_{n-2}"] & \cdots & P_0 \arrow[r] \arrow[d, "f_0"] & P_{-1} \arrow[d, "f_{-1}"] \\
A_n \arrow[r, shift left=4.5] \arrow[r, dotted, shift left=1.5] \arrow[r, dotted, shift right=1.5] \arrow[r, shift right=4.5] & A_{n-1} \arrow[r, shift left=3] \arrow[r, dotted] \arrow[r, shift right=3] & A_{n-2} & \cdots & A_0 \arrow[r] & A_{-1}
\end{tikzcd}$$

For the second part of the proof, by lemma 3.14, we have to construct a semi-simplicial map $E(\mathbb{P}) \xrightarrow{h} E(\mathbb{A})^I$ such that $\epsilon_0(E(\mathbb{A}))h = E(f)$ and $\epsilon_1(E(\mathbb{A}))h = E(g)$. To do so, let's construct, by induction on $n$, morphisms $P_n \xrightarrow{h_i^n} A_{n+1}$ for $n \in \mathbb{N}$ and $i \in [n]$ such that $\partial_0 h_0^n = f_n$, $\partial_{n+1} h_n^n = g_n$ and

$$\partial_i h_j^n = \begin{cases} h_{j-1}^{n-1} \partial_i & \text{if } 0 \leqslant i < j \leqslant n \\ \partial_i h_{i-1}^n & \text{if } 0 < i = j \leqslant n \\ h_j^{n-1} \partial_{i-1} & \text{if } 0 < j+1 < i \leqslant n+1. \end{cases}$$

If $n = 0$, we know that $\partial_0 g_0 = f_{-1}\partial_0 = \partial_0 f_0$. Thus, lemma 3.20 gives us $P_0 \xrightarrow{h_0^0} A_1$ such that $\partial_0 h_0^0 = f_0$ and $\partial_1 h_0^0 = g_0$.

Now, suppose all the $h_i^r$'s have been constructed for $0 \leqslant r \leqslant n-1$, $n \geqslant 1$ and that they commute as expected with the $\partial_i$'s. We are going to construct the $h_i^n$'s. To do it, we construct $h_i^n$ by induction on $i \in [n]$ in such a way that it satisfies all the desired equalities where, of course, only already defined maps appear.

For $i = 0$, let $x_0 = f_n$ and $x_i = h_0^{n-1}\partial_{i-1}$ for $1 < i \leqslant n+1$. Let's check that this is a $(n+1, 1)$-$P_n$-horn. If $1 < j \leqslant n+1$, $\partial_0 h_0^{n-1}\partial_{j-1} = f_{n-1}\partial_{j-1} = \partial_{j-1}f_n$. Whereas, if $1 < i < j \leqslant n+1$, $\partial_i h_0^{n-1}\partial_{j-1} = h_0^{n-2}\partial_{i-1}\partial_{j-1} = h_0^{n-2}\partial_{j-2}\partial_{i-1} = \partial_{j-1}h_0^{n-1}\partial_{i-1}$. Thus, since $\mathbb{A}$ is $P_n$-Kan, we have a filler $P_n \xrightarrow{h_0^n} A_{n+1}$ for this horn. This satisfies $\partial_0 h_0^n = f_n$ and $\partial_i h_0^n = h_0^{n-1}\partial_{i-1}$ for $2 \leqslant i \leqslant n+1$. These are all the equalities involving $h_0^n$ which are already defined.

Now, suppose that $n \geqslant 2$ and that the $h_i^n$'s have been defined for $0 \leqslant i \leqslant l-1$ where $1 \leqslant l \leqslant n-1$ and let us construct $h_l^n$ (we shall construct $h_n^n$ for $n \geqslant 1$ afterwards). Set $y_i = h_{l-1}^{n-1}\partial_i$ for $0 \leqslant i < l$, $y_l = \partial_l h_{l-1}^n$ and $y_i = h_l^{n-1}\partial_{i-1}$ for $l+1 < i \leqslant n+1$. Let check that defines a $(n+1, l+1)$-$P_n$-horn.

If $0 \leqslant i < j < l$: $\partial_i h_{l-1}^{n-1}\partial_j = h_{l-2}^{n-2}\partial_i\partial_j = h_{l-2}^{n-2}\partial_{j-1}\partial_i = \partial_{j-1}h_{l-1}^{n-1}\partial_i$.

If $0 \leqslant i < l-1 < j = l$: $\partial_i\partial_l h_{l-1}^n = \partial_{l-1}\partial_i h_{l-1}^n = \partial_{l-1}h_{l-2}^{n-1}\partial_i = \partial_{l-1}h_{l-1}^{n-1}\partial_i$.

If $i = l-1$, $j = l$ and $l > 1$:

$$\begin{aligned}
\partial_{l-1}\partial_l h_{l-1}^n &= \partial_{l-1}\partial_{l-1}h_{l-1}^n \\
&= \partial_{l-1}\partial_{l-1}h_{l-2}^n \\
&= \partial_{l-1}\partial_l h_{l-2}^n \\
&= \partial_{l-1}h_{l-2}^{n-1}\partial_{l-1} \\
&= \partial_{l-1}h_{l-1}^{n-1}\partial_{l-1}.
\end{aligned}$$

If $l = 1$, $i = 0$ and $j = 1$: $\partial_0\partial_1 h_0^n = \partial_0\partial_0 h_0^n = \partial_0 f_n = f_{n-1}\partial_0 = \partial_0 h_0^{n-1}\partial_0$.

If $0 \leqslant i < l$ and $l+1 < j \leqslant n+1$:

$$\partial_i h_l^{n-1}\partial_{j-1} = h_{l-1}^{n-2}\partial_i\partial_{j-1} = h_{l-1}^{n-2}\partial_{j-2}\partial_i = \partial_{j-1}h_{l-1}^{n-1}\partial_i.$$

If $i = l$ and $l+1 < j \leqslant n+1$: $\partial_l h_l^{n-1}\partial_{j-1} = \partial_l h_{l-1}^{n-1}\partial_{j-1} = \partial_l\partial_j h_{l-1}^n = \partial_{j-1}\partial_l h_{l-1}^n$.

If $l+1 < i < j \leqslant n+1$: $\partial_i h_l^{n-1}\partial_{j-1} = h_l^{n-2}\partial_{i-1}\partial_{j-1} = h_l^{n-2}\partial_{j-2}\partial_{i-1} = \partial_{j-1}h_l^{n-1}\partial_{i-1}$.

Therefore, since $\mathbb{A}$ is $P_n$-Kan, we have a filler $P_n \xrightarrow{h_l^n} A_{n+1}$ for this $P_n$-horn. It satisfies $\partial_i h_l^n = h_{l-1}^{n-1}\partial_i$ for $0 \leqslant i < l$, $\partial_l h_l^n = \partial_l h_{l-1}^n$ and $\partial_i h_l^n = h_l^{n-1}\partial_{i-1}$ for $l+1 < i \leqslant n+1$. These are the only equalities involving $h_l^n$ which are already defined.

To concludes the induction, it remains to construct $h_n^n$. Let $z_i = h_{n-1}^{n-1}\partial_i$ for $i \in [n-1]$, $z_n = \partial_n h_{n-1}^n$ and $z_{n+1} = g_n$. We want to use lemma 3.20. So, let us check the hypothesis:

If $0 \leqslant i < j < n$: $\partial_i h_{n-1}^{n-1}\partial_j = h_{n-2}^{n-2}\partial_i\partial_j = h_{n-2}^{n-2}\partial_{j-1}\partial_i = \partial_{j-1}h_{n-1}^{n-1}\partial_i$.

If $0 \leqslant i < n-1 < j = n$: $\partial_i\partial_n h_{n-1}^n = \partial_{n-1}\partial_i h_{n-1}^n = \partial_{n-1}h_{n-2}^{n-1}\partial_i = \partial_{n-1}h_{n-1}^{n-1}\partial_i$.

If $i = n-1$, $j = n$ and $n > 1$:

$$\begin{aligned}
\partial_{n-1}\partial_n h_{n-1}^n &= \partial_{n-1}\partial_{n-1}h_{n-1}^n \\
&= \partial_{n-1}\partial_{n-1}h_{n-2}^n \\
&= \partial_{n-1}\partial_n h_{n-2}^n \\
&= \partial_{n-1}h_{n-2}^{n-1}\partial_{n-1} \\
&= \partial_{n-1}h_{n-1}^{n-1}\partial_{n-1}.
\end{aligned}$$

If $n = 1$, $i = 0$ and $j = 1$: $\partial_0\partial_1 h_0^1 = \partial_0 f_1 = f_0\partial_0 = \partial_0 h_0^0\partial_0$.

If $0 \leqslant i < n < j = n+1$: $\partial_i g_n = g_{n-1}\partial_i = \partial_n h_{n-1}^{n-1}\partial_i$.

If $i = n$ and $j = n+1$: $\partial_n g_n = g_{n-1}\partial_n = \partial_n h_{n-1}^{n-1}\partial_n = \partial_n\partial_{n+1}h_{n-1}^n = \partial_n\partial_n h_{n-1}^n$.

Therefore, by lemma 3.20, there exists a map $P_n \xrightarrow{h_n^n} A_{n+1}$ such that $\partial_i h_n^n = h_{n-1}^{n-1}\partial_i$

for $0 \leqslant i < n$, $\partial_n h_n^n = \partial_n h_{n-1}^n$ and $\partial_{n+1} h_n^n = g_n$ which are the desired equalities. So the induction is completed.

Now we are going to construct $E(\mathbb{P}) \xrightarrow{h} E(\mathbb{A})^I$ . Let $h_0 = E(h_0^0) \in \mathcal{C}(E(P_0), E(\mathbb{A})_0^I)$ and for $n \geqslant 1$, $E(P_n) \xrightarrow{h_n} E(\mathbb{A})_n^I$ is the morphism defined by $\mathrm{pr}_i h_n = E(h_{i-1}^n)$ for each $i \in \{1, \ldots, n+1\}$. Note that this morphism is well-defined since for all $i \in \{1, \ldots, n\}$, $E(\partial_i)E(h_{i-1}^n) = E(\partial_i)E(h_i^n)$. Let's check that $h$ is a semi-simplicial map: first, notice that $\partial_0^I h_1 = E(\partial_0)E(h_1^1) = E(h_0^0)E(\partial_0) = h_0 E(\partial_0)$ and $\partial_1^I h_1 = E(\partial_2)E(h_0^1) = E(h_0^0)E(\partial_1) = h_0 E(\partial_1)$. Then, we have to prove that $\partial_i^I h_{n+1} = h_n E(\partial_i)$ for $n \geqslant 1$ and $i \in [n+1]$. To do so, it is enough to remark that

$$\mathrm{pr}_j \partial_i^I h_{n+1} = E(\partial_{i+1})\mathrm{pr}_j h_{n+1} = E(\partial_{i+1})E(h_{j-1}^{n+1}) = E(h_{j-1}^n)E(\partial_i) = \mathrm{pr}_j h_n E(\partial_i)$$

for all $1 \leqslant j \leqslant i$ and that

$$\mathrm{pr}_j \partial_i^I h_{n+1} = E(\partial_i)\mathrm{pr}_{j+1} h_{n+1} = E(\partial_i)E(h_j^{n+1}) = E(h_{j-1}^n)E(\partial_i) = \mathrm{pr}_j h_n E(\partial_i)$$

for all $i < j \leqslant n+1$. It remains to show that $\epsilon_0(E(\mathbb{A}))h = E(f)$ and $\epsilon_1(E(\mathbb{A}))h = E(g)$. But it follows from definition: $\epsilon_0(E(\mathbb{A}))_n h_n = E(\partial_0)E(h_0^n) = E(f_n)$ and $\epsilon_1(E(\mathbb{A}))_n h_n = E(\partial_{n+1})E(h_n^n) = E(g_n)$ for all $n \in \mathbb{N}$.

□

# 4 Comonadic Homology

In chapter 3, we studied simplicial objects. In this chapter, we are going to consider particular simplicial objects, the ones arising from a comonad. If we 'push' such simplicial objects forwards into a semi-abelian category with a functor, they induce a homology called comonadic homology. This is a functor from the initial category $\mathcal{D}$ to the semi-abelian one, $\mathcal{A}$. It can also be viewed as a functor from $[\mathcal{D},\mathcal{A}]$ to $[\mathcal{D},\mathcal{A}]$. However, in section 4.2, we shall focus on a necessary condition for two comonads to induce the same homology. Fortunately, all the hard part of the work has been accomplished in chapter 3, with the Comparison Theorem. The last section gives some examples of such homologies.

## 4.1 Comonads

Comonads are the duals of monads. They can be viewed as the information we get from an adjunction $F \dashv H$ where $F : \mathcal{C} \to \mathcal{D}$ and $H : \mathcal{D} \to \mathcal{C}$ without mentioning what happens on $\mathcal{C}$. In this section, we are constructing a simplicial object from a comonad.

**Definition 4.1.** A **comonad** $\mathbb{G} = (G, \varepsilon, \delta)$ in a category $\mathcal{D}$ consists of a functor $G : \mathcal{D} \to \mathcal{D}$ and two natural transformations $\varepsilon : G \to 1_{\mathcal{D}}$ and $\delta : G \to G^2$ making the diagrams

$$\begin{tikzcd} G \arrow[r, "\delta"] \arrow[dr, "1_G"'] & G^2 \arrow[d, "G\varepsilon"] \\ & G \end{tikzcd} \tag{4.1}$$

$$\begin{tikzcd} G \arrow[r, "\delta"] \arrow[dr, "1_G"'] & G^2 \arrow[d, "\varepsilon_G"] \\ & G \end{tikzcd} \tag{4.2}$$

$$\begin{tikzcd} G \arrow[r, "\delta"] \arrow[d, "\delta"'] & G^2 \arrow[d, "\delta_G"] \\ G^2 \arrow[r, "G\delta"'] & G^3 \end{tikzcd} \tag{4.3}$$

commute. We call $\varepsilon$ the **counit**, while $\delta$ is the **comultiplication**. Diagrams (4.1) and (4.2) are called the **counit laws**, whereas (4.3) is the **coassociativity law**.

**Example 4.2.** Let $F \dashv H$ be an adjunction where $F : \mathcal{C} \to \mathcal{D}$ and $H : \mathcal{D} \to \mathcal{C}$. Let $\eta : 1_{\mathcal{C}} \to HF$ and $\varepsilon : FH \to 1_{\mathcal{D}}$ be respectively the unit and counit of the adjunction. If we set $G = FH$ and $\delta = F\eta_H : G \to G^2$, then $\mathbb{G} = (G, \varepsilon, \delta)$ is a comonad of $\mathcal{D}$. Indeed, the counit laws follows from triangular identities and the coassociativity law from naturality of $\eta$.

The following lemma encompasses all the simplicial identities we have to prove in order to construct a simplicial object.

**Lemma 4.3.** If we have a comonad $\mathbb{G}$ in $\mathcal{D}$ and an object $A \in \text{ob } \mathcal{D}$, for all $n \in \mathbb{N}$, we have the following identities:

$$G^i\varepsilon_{G^{n-i}A} \circ G^j\varepsilon_{G^{n+1-j}A} = G^{j-1}\varepsilon_{G^{n-j+1}A} \circ G^i\varepsilon_{G^{n-i+1}A} \quad \forall\, 0 \leqslant i < j \leqslant n+1, \tag{4.4}$$

$$G^i\delta_{G^{n+1-i}A} \circ G^j\delta_{G^{n-j}A} = G^{j+1}\delta_{G^{n-j}A} \circ G^i\delta_{G^{n-i}A} \quad \forall\, 0 \leqslant i \leqslant j \leqslant n, \tag{4.5}$$

$$G^i\varepsilon_{G^{n+1-i}A} \circ G^j\delta_{G^{n-j}A} = G^{j-1}\delta_{G^{n-j}A} \circ G^i\varepsilon_{G^{n-i}A} \quad \forall\, 0 \leqslant i < j \leqslant n, \tag{4.6}$$

$$G^i\varepsilon_{G^{n+1-i}A} \circ G^j\delta_{G^{n-j}A} = 1_{G^{n+1}A} \quad \forall\, 0 \leqslant i = j \leqslant n \text{ and } \forall\, 1 \leqslant i = j+1 \leqslant n, \tag{4.7}$$

$$G^i\varepsilon_{G^{n+1-i}A} \circ G^j\delta_{G^{n-j}A} = G^j\delta_{G^{n-j-1}A} \circ G^{i-1}\varepsilon_{G^{n-i+1}A} \quad \forall\, 1 \leqslant j+1 < i \leqslant n. \tag{4.8}$$

*Proof.* Let's prove the first identity:

$$\begin{aligned} G^i\varepsilon_{G^{n-i}A} \circ G^j\varepsilon_{G^{n+1-j}A} &= G^i(\varepsilon_{G^{n-i}A} \circ G^{j-i}\varepsilon_{G^{n+1-j}A}) \\ &= G^i(G^{j-1-i}\varepsilon_{G^{n-j+1}A} \circ \varepsilon_{G^{n-i+1}A}) \\ &= G^{j-1}\varepsilon_{G^{n-j+1}A} \circ G^i\varepsilon_{G^{n-i+1}A} \end{aligned}$$

where we used the naturality of $\varepsilon$.
For the second one, we also do a direct computation:

$$\begin{aligned} G^i\delta_{G^{n+1-i}A} \circ G^j\delta_{G^{n-j}A} &= G^i(\delta_{G^{n+1-i}A} \circ G^{j-i}\delta_{G^{n-j}A}) \\ &= G^i(G^{j+1-i}\delta_{G^{n-j}A} \circ \delta_{G^{n-i}A}) \\ &= G^{j+1}\delta_{G^{n-j}A} \circ G^i\delta_{G^{n-i}A} \end{aligned}$$

where we used the naturality of $\delta$ if $i < j$ or the coassociativity law (4.3) if $i = j$.
For the third identity, we use again the naturality of $\varepsilon$:

$$\begin{aligned} G^i\varepsilon_{G^{n+1-i}A} \circ G^j\delta_{G^{n-j}A} &= G^i(\varepsilon_{G^{n+1-i}A} \circ G^{j-i}\delta_{G^{n-j}A}) \\ &= G^i(G^{j-i-1}\delta_{G^{n-j}A} \circ \varepsilon_{G^{n-i}A}) \\ &= G^{j-1}\delta_{G^{n-j}A} \circ G^i\varepsilon_{G^{n-i}A}. \end{aligned}$$

To prove (4.7) when $i = j$ we use the second counit law (4.2)

$$G^i\varepsilon_{G^{n+1-i}A} \circ G^i\delta_{G^{n-i}A} = G^i(\varepsilon_{G^{n+1-i}A} \circ \delta_{G^{n-i}A}) = G^i(1_{G^{n-i+1}A}) = 1_{G^{n+1}A},$$

while we use the first counit law (4.1) if $i = j+1$

$$G^{j+1}\varepsilon_{G^{n-j}A} \circ G^j\delta_{G^{n-j}A} = G^j(G\varepsilon_{G^{n-j}A} \circ \delta_{G^{n-j}A}) = G^j(1_{G^{n-j+1}A}) = 1_{G^{n+1}A}.$$

Finally, we prove the last identity using the naturality of $\delta$:

$$\begin{aligned} G^i\varepsilon_{G^{n+1-i}A} \circ G^j\delta_{G^{n-j}A} &= G^j(G^{i-j}\varepsilon_{G^{n+1-i}A} \circ \delta_{G^{n-j}A}) \\ &= G^j(\delta_{G^{n-j-1}A} \circ G^{i-1-j}\varepsilon_{G^{n-i+1}A}) \\ &= G^j\delta_{G^{n-j-1}A} \circ G^{i-1}\varepsilon_{G^{n-i+1}A}. \end{aligned}$$

$\square$

Due to this lemma, $\mathbb{G}$ induces an augmented simplicial object in $\mathcal{D}$.

**Definition 4.4.** Let $\mathbb{G}$ be a comonad in $\mathcal{D}$ and $A \in \text{ob } \mathcal{D}$. If we write, for all $n \in \mathbb{N}$ and $i \in [n]$, $\partial_i$ for the map $G^{n+1}A \xrightarrow{G^i\varepsilon_{G^{n-i}A}} G^nA$, $\sigma_i$ for $G^{n+1}A \xrightarrow{G^i\delta_{G^{n-i}A}} G^{n+2}A$ and $A_n = G^{n+1}A$ for all $n \geqslant -1$, then $(A_n)_{n\geqslant -1}$ is an augmented simplicial object in $\mathcal{D}$. We denote it by $\mathbb{G}A$.

$$\begin{tikzcd} \cdots \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & G^3A \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & G^2A \arrow[r, shift left] \arrow[r, shift right] & GA \arrow[r] & A \end{tikzcd}$$

Moreover, if $f \in \mathcal{D}(A, B)$, we have an induced morphism $\mathbb{G}(f) \in \mathrm{AS}\,\mathcal{D}(\mathbb{G}A, \mathbb{G}B)$ by setting $(\mathbb{G}(f))_n = G^{n+1}f$ for all $n \geqslant -1$. This is a well-defined morphism in the category $\mathrm{AS}\,\mathcal{D}$ since $(\mathbb{G}(f))_{n-1} \circ \partial_i = \partial'_i \circ (\mathbb{G}(f))_n$ and $(\mathbb{G}(f))_n \circ \sigma_i = \sigma'_i \circ (\mathbb{G}(f))_{n-1}$ by naturality of $\varepsilon$ and $\delta$. This makes $\mathbb{G}$ into a functor $\mathbb{G} : \mathcal{D} \to \mathrm{AS}\,\mathcal{D}$.

If $\mathcal{D}$ is semi-abelian, we can define the homology sequence of the Moore complex of $\mathbb{G}A$. This is what we are going to do in the next section.

## 4.2 Comonadic homology

Recall that if we have a functor $E : \mathcal{D} \to \mathcal{A}$, it can be turned into a functor $E : \mathrm{S}\,\mathcal{D} \to \mathrm{S}\,\mathcal{A}$ in the obvious way. This is how $\mathbb{G}A$ can induce a simplicial object (and so a homology) in a semi-abelian category. The main result of this section, and of this essay, says that, provided some condition on two comonads $\mathbb{G}$ and $\mathbb{K}$ holds, $\mathbb{G}A$ and $\mathbb{K}A$ induce the same homology.

**Definition 4.5.** Let $\mathcal{A}, \mathcal{D}$ be two categories with $\mathcal{A}$ semi-abelian. Let also $\mathbb{G} = (G, \varepsilon, \delta)$ be a comonad in $\mathcal{D}$, $A$ be an object in $\mathcal{D}$ and $E : \mathcal{D} \to \mathcal{A}$ be a functor. For all $n \in \mathbb{Z}$,

$$H_n(A, E)_{\mathbb{G}} = H_{n-1}NE\mathbb{G}A \tag{4.9}$$

is called the **$\mathbf{n^{th}}$ homology object of $A$ (with coefficients in $E$) relative to the comonad** $\mathbb{G}$. This induces a functor $H_n(-, E)_{\mathbb{G}} : \mathcal{D} \to \mathcal{A}$. Moreover, if $E, E' : \mathcal{D} \to \mathcal{A}$ are two functors and $\alpha : E \to E'$ is a natural transformation, we can define another natural transformation $H_n(-, \alpha)_{\mathbb{G}} := \beta : H_n(-, E)_{\mathbb{G}} \to H_n(-, E')_{\mathbb{G}}$ by setting $\beta_A$ as the image by $H_{n-1}N$ of the augmented simplicial map

$$\begin{tikzcd}
\cdots \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & EG^3A \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d, "\alpha_{G^3A}"] & EG^2A \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\alpha_{G^2A}"] & EGA \arrow[r] \arrow[d, "\alpha_{GA}"] & EA \arrow[d, "\alpha_A"] \\
\cdots \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & E'G^3A \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & E'G^2A \arrow[r, shift left] \arrow[r, shift right] & E'GA \arrow[r] & E'A
\end{tikzcd}$$

This makes $H_n(-,\ )_{\mathbb{G}}$ be a functor $[\mathcal{D}, \mathcal{A}] \to [\mathcal{D}, \mathcal{A}]$.

Actually, we can already compute this homology for some particular objects of $\mathcal{D}$.

**Proposition 4.6.** Let $\mathcal{A}, \mathcal{D}$ be two categories with $\mathcal{A}$ semi-abelian, $\mathbb{G}$ a comonad in $\mathcal{D}$, $E : \mathcal{D} \to \mathcal{A}$ a functor and $A \in \mathrm{ob}\ \mathcal{D}$. Then,

$$H_n(GA, E)_{\mathbb{G}} = \begin{cases} EGA & \text{if } n = 1 \\ 0 & \text{if } n \neq 1. \end{cases}$$

*Proof.* $\mathbb{A} = E\mathbb{G}GA$ is an augmented simplicial object in $\mathcal{A}$. It is actually right-contractible. Indeed, we can define $A_n = EG^{n+2}A \xrightarrow{h_n = EG^{n+1}\delta_A} EG^{n+3}A = A_{n+1}$ for all $n \geqslant -1$ which satisfies

$$\partial_{n+1}h_n = EG^{n+1}\varepsilon_{GA} \circ EG^{n+1}\delta_A = 1_{EG^{n+2}A} = 1_{A_n}$$

for all $n \geqslant -1$ by the counit law (4.2) and

$$\partial_i h_n = EG^i\varepsilon_{G^{n+2-i}A} \circ EG^{n+1}\delta_A = EG^n\delta_A \circ EG^i\varepsilon_{G^{n+1-i}A} = h_{n-1}\partial_i$$

for all $n \geqslant 0$ and $i \in [n]$ by naturality of $\varepsilon$. The result follows from proposition 3.8. Recall the dimension shift, i.e. $H_n(GA, E)_{\mathbb{G}} = H_{n-1}NE\mathbb{G}GA$.

□

A natural question that one can wonder about comonadic homology, is to know if two different comonads can give rise to the same homology. We are going to give a sufficient condition on these comonads to have the same homology. To do so, we need to introduce the projective class generated by a comonad (see definition 3.15 and remark 3.16).

**Definition 4.7.** Let $\mathbb{G}$ be a comonad in the category $\mathcal{D}$. **The projective class generated by** $\mathbb{G}$ is the projective class generated by the class $\{GA \mid A \in \text{ob } \mathcal{D}\}$. We denote it by $(\mathcal{P}_{\mathbb{G}}, \mathcal{E}_{\mathbb{G}})$.

**Remark 4.8.** To see that this is really a projective class, we have to check that $\mathcal{D}$ has enough $\mathcal{E}_{\mathbb{G}}$ projectives. Let $A \in \text{ob } \mathcal{D}$. We know that $GA \in \mathcal{P}_{\mathbb{G}}$. Moreover, $GA \xrightarrow{\varepsilon_A} A$ is in $\mathcal{E}_{\mathbb{G}}$ since, if we have $GB \xrightarrow{f} A$, it equals $GB \xrightarrow{Gf \circ \delta_B} GA \xrightarrow{\varepsilon_A} A$, by naturality of $\varepsilon$ and (4.2). Therefore $\mathcal{D}$ has enough $\mathcal{E}_{\mathbb{G}}$ projectives.

There is a better description of $\mathcal{P}_{\mathbb{G}}$.

**Lemma 4.9.** Let $\mathbb{G}$ be a comonad in $\mathcal{D}$. Then,

$$\mathcal{P}_{\mathbb{G}} = \{P \in \text{ob } \mathcal{D} \mid \exists s \in \mathcal{D}(P, GP) \text{ such that } \varepsilon_P s = 1_P\}.$$

*Proof.* Let $P \in \mathcal{P}_{\mathbb{G}}$. Since, $GP \xrightarrow{\varepsilon_P} P$ is in $\mathcal{E}_{\mathbb{G}}$, $1_P$ factors through it which gives $s \in \mathcal{D}(P, GP)$.

Conversely, suppose $P \in \text{ob } \mathcal{D}$ and $P \xrightarrow{s} GP$ are such that $\varepsilon_P s = 1_P$. Then, if $A \xrightarrow{e} B$ is in $\mathcal{E}_{\mathbb{G}}$, we have to prove that $P$ is $e$-projective. That is, given $P \xrightarrow{f} B$, we have to show that $f$ factors through $e$. But since $e$ is in $\mathcal{E}_{\mathbb{G}}$, there exists $h \in \mathcal{D}(GP, A)$ such that $eh = f\varepsilon_P$. Thus, $ehs = f\varepsilon_P s = f$ and $f$ factors through $e$.

□

Now we have everything we need to prove the main theorem of this essay.

**Theorem 4.10.** Let $\mathbb{G}$ and $\mathbb{K}$ be two comonads on a category $\mathcal{D}$ such that $\mathcal{P}_{\mathbb{G}} = \mathcal{P}_{\mathbb{K}}$ and this is a Kan projective class on $\mathcal{D}$. If $\mathcal{A}$ is a semi-abelian category and $E : \mathcal{D} \to \mathcal{A}$ a functor, then $H_n(-, E)_{\mathbb{G}}$ and $H_n(-, E)_{\mathbb{K}}$ are naturally isomorphic for all $n \in \mathbb{Z}$.

*Proof.* Set $\mathcal{P} = \mathcal{P}_{\mathbb{G}} = \mathcal{P}_{\mathbb{K}}$ and fix $A \in \text{ob } \mathcal{D}$. First, let's prove that $\mathbb{G}A$ is $\mathcal{P}_{\mathbb{G}}$-left-contractible. Let $P \in \mathcal{P}_{\mathbb{G}}$ and $s \in \mathcal{D}(P, GP)$ given by lemma 4.9. For all $n \geqslant -1$, define the mapping $h_n : \mathcal{C}(P, G^{n+1}A) \longrightarrow \mathcal{C}(P, G^{n+2}A) : f \mapsto Gf \circ s$. It satisfies the required condition since $\partial_0 Gfs = \varepsilon_{G^{n+1}A} Gfs = f\varepsilon_P s = f$ and $\partial_i Gfs = G^i \varepsilon_{G^{n+1-i}A} Gfs = G(G^{i-1}\varepsilon_{G^{n+1-i}A} f)s = G(\partial_{i-1} f)s$ for all $i \in \{1, \ldots, n+1\}$. Therefore $\mathbb{G}A$ is $\mathcal{P}$-left-contractible and also $\mathcal{P}$-Kan since $\mathcal{P}$ is a Kan projective class by assumption. Similarly, $\mathbb{K}A$ is $\mathcal{P}$-Kan and $\mathcal{P}$-left-contractible.

But, by definition of $\mathcal{P}$, $GB, KB \in \mathcal{P}$ for all $B \in \text{ob } \mathcal{D}$, so we can use the Comparison Theorem 3.21 to get two augmented semi-simplicial maps $\mathbb{G}A \xrightarrow{f^A} \mathbb{K}A$ and $\mathbb{K}A \xrightarrow{g^A} \mathbb{G}A$ such that $f^A_{-1} = g^A_{-1} = 1_A$.

$$\begin{tikzcd}
\cdots & G^{n+2}A \arrow[r, shift left=4.5] \arrow[r, dotted, shift left=1.5] \arrow[r, dotted, shift right=1.5] \arrow[r, shift right=4.5] \arrow[d, "f^A_{n+1}"'] & G^{n+1}A \arrow[r, shift left=3] \arrow[r, dotted] \arrow[r, shift right=3] \arrow[d, "f^A_n"] & G^nA \arrow[d, "f^A_{n-1}"] & \cdots & GA \arrow[r] \arrow[d, "f^A_0"] & A \arrow[d, "1_A"] \\
\cdots & K^{n+2}A \arrow[r, shift left=4.5] \arrow[r, dotted, shift left=1.5] \arrow[r, dotted, shift right=1.5] \arrow[r, shift right=4.5] \arrow[d, "g^A_{n+1}"'] & K^{n+1}A \arrow[r, shift left=3] \arrow[r, dotted] \arrow[r, shift right=3] \arrow[d, "g^A_n"] & K^nA \arrow[d, "g^A_{n-1}"] & \cdots & KA \arrow[r] \arrow[d, "g^A_0"] & A \arrow[d, "1_A"] \\
\cdots & G^{n+2}A \arrow[r, shift left=4.5] \arrow[r, dotted, shift left=1.5] \arrow[r, dotted, shift right=1.5] \arrow[r, shift right=4.5] & G^{n+1}A \arrow[r, shift left=3] \arrow[r, dotted] \arrow[r, shift right=3] & G^nA & \cdots & GA \arrow[r] & A
\end{tikzcd}$$

Moreover, by the second part of the Comparison Theorem 3.21, we know that

$$H_nNE(g^Af^A) = H_nNE(1_{\mathbb{G}A}) = 1_{H_nNE\mathbb{G}A}$$

and

$$H_nNE(f^Ag^A) = H_nNE(1_{\mathbb{K}A}) = 1_{H_nNE\mathbb{K}A}.$$

Therefore

$$H_{n+1}(A,E)_{\mathbb{G}} \cong H_{n+1}(A,E)_{\mathbb{K}}.$$

It remains to show that this isomorphism is natural in $A$. If we have $A \xrightarrow{h} B$ in $\mathcal{D}$, we can consider the two augmented semi-simplicial maps $\mathbb{G}A \xrightarrow{f^A} \mathbb{K}A \xrightarrow{\mathbb{K}h} \mathbb{K}B$ and $\mathbb{G}A \xrightarrow{\mathbb{G}h} \mathbb{G}B \xrightarrow{f^B} \mathbb{K}B$ . They satisfy $(\mathbb{K}h \circ f^A)_{-1} = h = (f^B \circ \mathbb{G}h)_{-1}$. Thus, by the Comparison Theorem 3.21, $H_nNE(\mathbb{K}h \circ f^A) = H_nNE(f^B \circ \mathbb{G}h)$, which proves the naturality of the isomorphism $H_nNE(f^A)$.

$$\begin{tikzcd}
H_{n+1}(A,E)_{\mathbb{G}} \arrow[r, "H_nNE\mathbb{G}h"] \arrow[d, "H_nNE(f^A)"'] & H_{n+1}(B,E)_{\mathbb{G}} \arrow[d, "H_nNE(f^B)"] \\
H_{n+1}(A,E)_{\mathbb{K}} \arrow[r, "H_nNE\mathbb{K}h"'] & H_{n+1}(B,E)_{\mathbb{K}}
\end{tikzcd}$$

□

## 4.3 Examples

The aim of this section is to exhibit some examples of comonadic homologies. Some of them are well-known homology theories. Of course, the following list is not exhaustive. For 'brevity' of this essay, the results of this section are left unproved. Let's start with the trivial example.

**Example 4.11** (Trivial)**.** If $\mathbb{G} = (1_{\mathcal{D}}, 1, 1)$, by proposition 4.6 or by direct computations, we have

$$H_n(A,E)_{\mathbb{G}} = \begin{cases} EA & \text{if } n = 1 \\ 0 & \text{if } n \neq 1. \end{cases}$$

The next one is a well-know homology in Commutative Algebra.

**Example 4.12** (Tor)**.** Let $R$ be a unitary commutative ring, $\mathcal{D} = R$-Mod and $\mathbb{G}$ be the comonad induced by the forgetful/free adjunction (where $\mathcal{C} = \text{Set}$). If $\mathcal{A} = R$-Mod and $E : R\text{-Mod} \to R\text{-Mod}$ is the functor given by $- \otimes_R N$ for a fixed $R$-module $N$, then, one can prove (see [5]) that

$$H_n(M, - \otimes_R N)_{\mathbb{G}} = \mathrm{Tor}^R_{n-1}(M,N).$$

Due to proposition 4.6, we have another proof of

$$\mathrm{Tor}^R_n(R^{\oplus M}, N) = \begin{cases} R^{\oplus M} \otimes_R N & \text{if } n = 0 \\ 0 & \text{if } n > 0. \end{cases}$$

Now, we give an example from Algebraic Topology.

**Example 4.13** (Singular Homology)**.** Let's construct a comonad in the category $\mathcal{D} = \mathrm{Top}$. We denote by $\Delta_p$ the $p$-dimensional simplex. Let $G$ be the functor

$$G : \mathrm{Top} \to \mathrm{Top} : X \to \bigsqcup_{\substack{\Delta_p \to X \\ p \geqslant 0}} \Delta_p$$

where the disjoint union is over all continuous map $\Delta_p \to X$. The action of $G$ on arrows is given by the following: if $g : X \to Y$ is a continuous function, then

$$Gg : x \in (\Delta_p)_f \mapsto x \in (\Delta_p)_{gf}.$$

The counit is the natural transformation

$$\varepsilon_X : GX \to X : x \in (\Delta_p)_f \longmapsto f(x)$$

while the comultiplication is given by

$$\delta_X : GX \to G^2X : x \in (\Delta_p)_f \longmapsto x \in (\Delta_p)_{(\Delta_p)_f \hookrightarrow GX}.$$

It is easy to check that $\mathbb{G} = (G, \varepsilon, \delta)$ is a comonad.

Now, if $\mathcal{A} = \mathrm{AbGp}$ and $E = H_0^{\mathrm{sing}} : \mathrm{Top} \to \mathrm{AbGp}$ is the $0^{\mathrm{th}}$ singular homology functor, Barr and Beck proved in [5] that $H_n(X, H_0^{\mathrm{sing}})_{\mathbb{G}}$ is the $n-1^{\mathrm{th}}$ singular homology group of $X$.

The last example says that in preadditive categories, we do not have to prove that the projective class is Kan to use theorem 4.10.

**Example 4.14.** Moore showed in [11] that if $\mathcal{D}$ is a preadditive category, then, every simplicial object is $X$-Kan for all $X \in \mathrm{ob}\ \mathcal{D}$. Thus every projective class is Kan and we can rewrite theorem 4.10 as:

Let $\mathbb{G}$ and $\mathbb{K}$ be two comonads on a preadditive category $\mathcal{D}$ such that $\mathcal{P}_{\mathbb{G}} = \mathcal{P}_{\mathbb{K}}$. If $\mathcal{A}$ is a semi-abelian category and $E : \mathcal{D} \to \mathcal{A}$ a functor, then $H_n(-, E)_{\mathbb{G}}$ and $H_n(-, E)_{\mathbb{K}}$ are naturally isomorphic for all $n \in \mathbb{Z}$.

# 5 Conclusion

Homology was first studied to count the number of holes in a topological space. Now, it has many other applications in Mathematics. A natural way to generalise it is to use Category Theory. Classically, homology is studied in abelian categories such as AbGp or $R$-Mod. We exhibited in this essay another context where it can be done: semi-abelian categories. As we have seen, the definition of semi-abelian category is less restrictive than the one of abelian category in order to encompass examples such as Gp and LieAlg. However, there are sufficiently many axioms to have an image factorisation, to prove their finite completeness and cocompleteness and to let the Five, Nine and Snake lemmas hold.

In this essay, we focused on a particular kind of homology: the one arising from a simplicial object and especially a simplicial object made from a comonad. This is called the comonadic homology. The main result of this essay gives a condition on two comonads to induce the same homology. But there is much more to say about comonadic homology. Indeed, we could have focused this essay on the functorial dependence of $H_n(-, E)_{\mathbb{G}}$ in $E$ and its properties of $\mathbb{G}$-acyclicity and $\mathbb{G}$-connectedness. We could also have compared it to the case where $\mathcal{A}$ is abelian. With this additional hypothesis, Barr and Beck proved in [5] that we no longer need the Kan condition in theorem 4.10.

Comonadic homology is not the only way to define a homology in semi-abelian categories. Indeed, we can extend the well-known Hopf formula to higher dimensions to define a homology theory in semi-abelian categories. We can prove (see [1]) that they coincide whenever they are both defined, which gives two different ways to study the same object. Two other definitions of homology are possible in the semi-abelian context. One uses Galois groupoids while the other one is constructed from a satellite. For more details, we refer to [1].

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